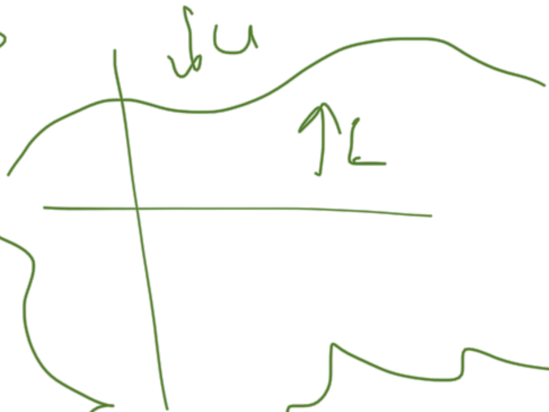
NOTES, HINTS, AND SOLUTIONS HW12.

problem 1.

$$f \in \mathcal{R} \text{ over } [a,b] \iff \begin{cases} f \in \mathcal{R} \text{ over } [a,x_0] \\ \sim \quad \sim \quad [x_0,b] \end{cases}$$

use lemma B.3:

$f \in \mathcal{R}$ over $[a,b] \iff \forall \varepsilon>0, \exists P$ s.t.
partition of $[a,b]$



$(*)$ $U(f,P) - L(P,f) \leq \varepsilon$

- for $(\Rightarrow)$. Let P be the partition s.t. $*$ holds and let P^* be the partition P after adding x_0 to it.

- show that $(*)$ holds for P^* and explain that there is a one to one correspondence between the pair of partitions of $[a,x_0]$ and $[x_0,b]$ and partitions of $[a,b]$ containing x_0.

- use this to show that $(*)$ holds for some partition of $[a, x_0]$ and $[x_0, b]$.

for $(\Longleftarrow)$: if P is a partition of $[a, x_0]$ and P' is a partition of $[x_0, b]$ s.t. for both $(*)$ holds, find the partition P^* for $[a, b]$ for which $(*)$ holds.

To show the equality $\int_a^b f = \int_a^{x_0} f + \int_{x_0}^b f$

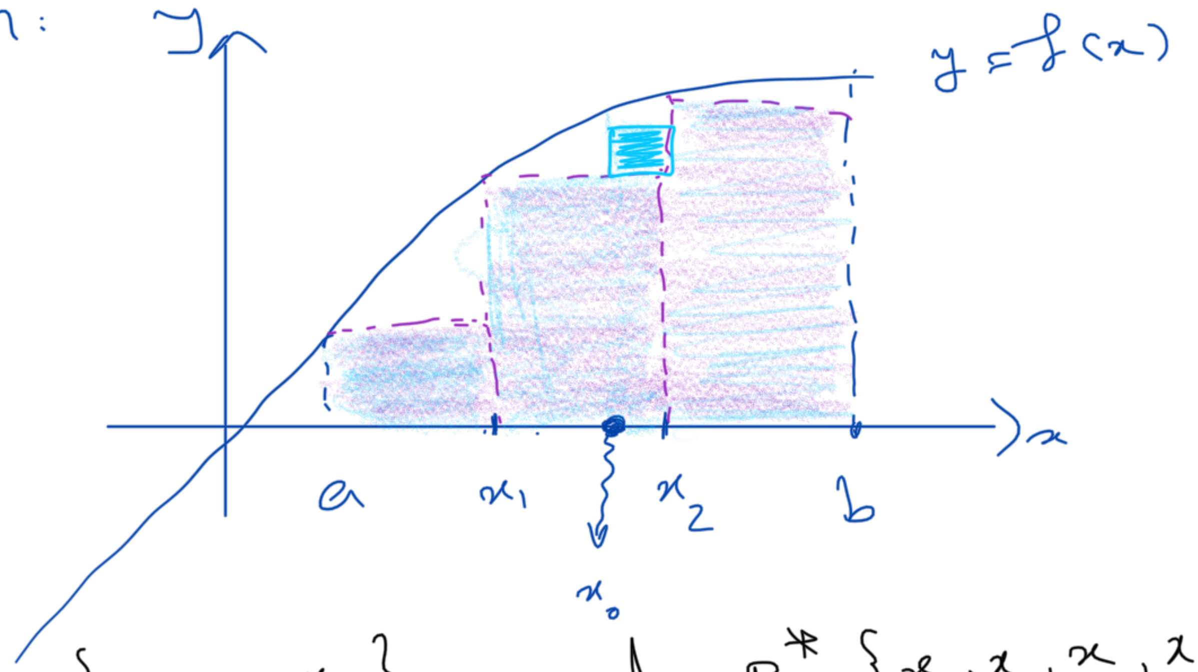
I suggest that you show $\leq$ and $\geq$.

to show the inequality one way is to use the definition $\int_a^b f = \sup_{\substack{\text{over all} \\ \text{partitons } P \\ \text{of } [a,b]}} L(f, P)$. and the one to one correspondence that is mentioned before.

Remark: $A \subseteq B$ $\quad \sup A \leq \sup B$

$\inf A \geq \inf B$

Solution:



Let $P = \{x_1, x_2, x_3\}$ and $P^* = \{x_1, x_0, x_2, x_3\}$

$L(f, P)$ is shown by purple.

$L(f, P^*)$ is shown by blue.

Now for <u>any</u> partition P of $[a,b]$, Let P^* be the partition $P \cup \{x_0\}$.

As you can see, by taking more partition points our estimation gets better, meaning that $L(f,P)$ increases and $U(f,P)$ dicreases. Therefore, for any partition P' that $P \subseteq P'$ we have:

$$U(f,P') - L(f,P') \leqslant U(f,P) - L(f,P). \quad (*)$$

In particular, this holds for $P' = P^*$

Now let $f \in \mathcal{R}$ over $[a,b]$. This means:

$\forall \varepsilon > 0 \ \exists$ partition P such that $U(f,P) - L(f,P) < \varepsilon$

from $(*)$ you have $U(f,P^*) - L(f,P^*) \leq U(f,P) - L(f,P)$

Therefore, $\forall \varepsilon > 0 \ \exists$ partition P^* such that:

$$U(f, P^*) - L(f, P^*) \leq \varepsilon \qquad (**)$$

Let P_a^* be the partion P^* from a to x_0.

" P_b^* " " " " " x_0 to b.

we have: $U(f, P^*) = U(f, P_a^*) + U(f, P_b^*)$

and $L(f, P^*) = L(f, P_a^*) + L(f, P_b^*)$

By substituting them in $(**)$ we get:

$$\underbrace{\left[U(f, P_a^*) - L(f, P_a^*)\right]}_{\geq 0} + \underbrace{\left[U(f, P_b^*) - L(f, P_b^*)\right]}_{\geq 0} \leq \varepsilon$$

$\Rightarrow \forall \varepsilon > 0 \quad \exists P_a^*$ and P_b^* s.t.

$$U(f, P_a^*) - L(f, P_a^*) \leq \varepsilon$$

and $$U(f, P_b^*) - L(f, P_b^*) \leq \varepsilon$$

$\Rightarrow f \in \mathcal{R}$ over $[a, x_0]$ and $f \in \mathcal{R}$ over $[x_0, b]$.

Now let $f \in R$ over $[a, x_0]$ and $f \in R$ over $[x_0, b]$

This means $\forall \varepsilon > 0 \quad \exists$ Partition P_a of $[a, x_0]$ s.t.

$$U(f, P_a) - L(f, P_a) \leq \varepsilon/2$$

and $\exists$ partition P_b of $[x_0, b]$ such that:

$$U(f, P_b) - L(f, P_b) \leq \varepsilon/2$$

Now let $P = P_a \cup P_b$ be a partition of $[a, b]$. Similar to before we have

$$U(f,P) - L(f,P) = \Big[U(f,P_a) + U(f,P_b)\Big] - \Big[L(f,P_a) + L(f,P_b)\Big]$$

$$= \underbrace{\Big[U(f,P_a) - L(f,P)\Big]}_{\leq \varepsilon/2} + \underbrace{\Big[U(f,P_b) - L(f,P_b)\Big]}_{\leq \varepsilon/2} \leq \varepsilon$$

$\Rightarrow f \in \mathcal{R}$ over $[a,b]$.

By definition we have: If $f \in \mathcal{R}$ over $[\alpha,\beta]$,

$$\int_\alpha^\beta f := \sup_{\substack{\text{over all partitions} \\ P \text{ of } [\alpha,\beta]}} L(f,P)$$

now we know that $f \in \mathcal{R}$ over $[a,b]$, $[a,x_0]$, and $[x_0,b]$.

and we WTS: $\int_a^b f = \int_a^{x_0} f + \int_{x_0}^b f$.

The key point is the following:

If $f \in \mathcal{R}$ over $[\alpha,\beta]$, then

$$\sup_{\substack{\text{over \underline{all} partitions } P \\ \text{of } [\alpha,\beta]}} L(f,P) = \sup_{\substack{\text{over \underline{all partitions containing}} \\ \text{\underline{a point } } \gamma \in [\alpha,\beta]}} L(f,P) \quad \text{I}$$

(or more generally a finite number of points.)

to show I you can show both inequalities $\leq$ and $\geq$.

for $\leq$: to any partition P from the LHS you can add the point γ it gives you some partition P' from the RHS s.t. $L(f,p) \leq L(f,p')$. Thus $\leq$ holds.

for $\geq$: Note that if $B \subseteq A$ then $\sup_A F \geq \sup_B F$.

To complete the proof, remember the one to one correspondence between the partitions of $[a,b]$ containing x_0 and the pair of partitions P_a and P_b of $[a,x_0]$ respectively.

$$\underset{\substack{\text{over all partitions}\\ \text{of } [a,b] \text{ containing } x_0}}{\sup L(f,p)} = \underset{\substack{\text{over all partions}\\ P_a \text{ of } [a,x_0]}}{\sup L(f,P_a)} + \underset{\substack{\text{over all partitions}\\ P_b \text{ of } [x_0,b]}}{\sup L(f,P_a)}$$

call it II. From I and II you have the equality.

Problem 2.

Ex 1. Show that if $f:[a,b] \longrightarrow \mathbb{R}$ is integrable then f is bounded.

Ex 2. Show that $\forall$ partition P of $[a,b]$, if $f \in \mathcal{R}$ over $[a,b]$ and $g = \max(f,0)$ then

$$U(f,P) - L(f,P) \geq U(g,P) - L(g,P)$$

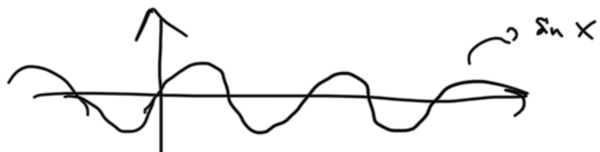
For 2(a) use Ex1 and theorem B.1.5.2.

For 2(b) use Ex2 and Lemma B.3.

solution: (a) well Ex1 is just true by the definition Therefore, if $f \in \mathcal{R}$ over $[a,b]$, f is bounded.

Theorem 1.5.2. If $f \in \mathcal{R}$ over $[a,b]$ (which means it is bounded by definition) and h is a continuous function on $Im(f)$ then the compositin $h(f)$ or $h \circ f$ is also integrable over $[a,b]$.

Since you know that the function $h = \sin x$ is continuous everywhere on $\mathbb{R}$



and f is integrable, there for $h \circ f = \sin(f)$ is integrable.

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b) for This you can use the same argument as before Since $g(x) = \min(x, 0)$ is a continuous function $\Rightarrow \min(f,0) \in \mathcal{R}$

$\boxed{\text{Or}}$ you can show that for any partition P of $[a,b]$ we have

$$(*)\quad U(\min(f,0),P) - L(\min(f,0),P) \leq U(f,P) - L(f,P)$$

now using Lemma B.3.

$$\forall \varepsilon > 0 \quad \exists P \text{ s.t. } U(\min(f,0),P) - L(\min(f,0),P) \leq \varepsilon$$

$$\Longrightarrow \min(f,0) \in \mathcal{R}.$$

It remains to show $(*)$

Note that $U(f,P) - L(f,P) = \sum_{i=1}^{n} r_i$, where

r_i = the area of the difference of boxes. i.e.

instead of adding up all the upper boxes and adding up all the lower boxes and then taking the difference, you can take the differences for each small interval and add them up. Therefore, $r_i \geq 0 \quad \forall i \in \{1, \dots, n\}$

each $r_i = u_i - l_i$ where u_i, l_i where $l_i \leq u_i$

upper box ↵ ↳ lower box

There are three cases ① $l_i \geq 0 \quad (\Rightarrow u_i \geq 0)$

② $l_i < 0 \qquad u_i > 0$

③ $l_i < 0 \qquad u_i \leq 0$

The claim is that

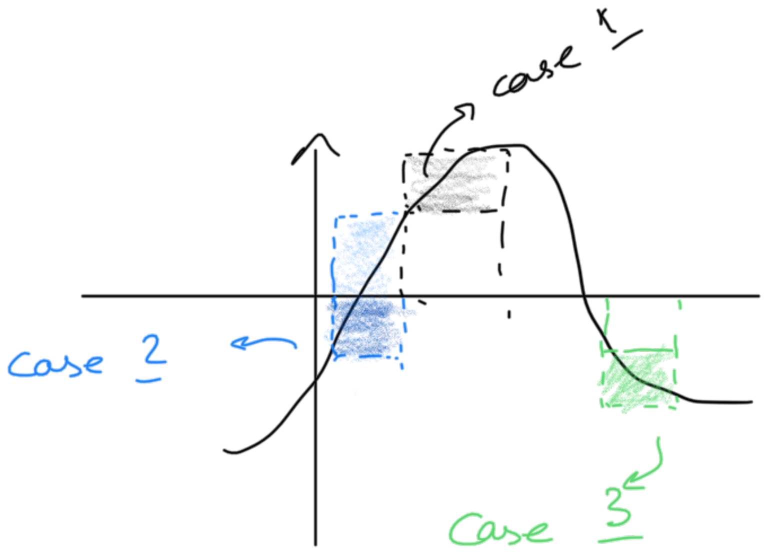
$$U(\min(f,0), P) - L(\min(f,0), P) = \sum_{i=1}^{n} s_i$$

where $s_i = 0$ in case ①

and $s_i < r_i$ in case ②

and $s_i = r_i$ in case ③

therefore $\textcircled{*}$ is true. See the cases in this picture ⇝



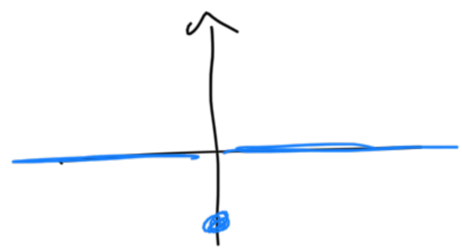
Problem 3.

Ex 3. let $f(x) = \begin{cases} 1 & x=-1 \\ 2 & x=0 \\ 3 & x=1 \\ 0 & \text{else} \end{cases}$

show that $f \in R$ over $[a,b]$ for any $a, b \in \mathbb{R}$ and $a<b$. Compute $\int_a^b f$.

problem 3,(a) is similar to Ex 3.

3 (a) Solution. $f = \begin{cases} -1 & x=0 \\ 0 & x \neq 0 \end{cases}$



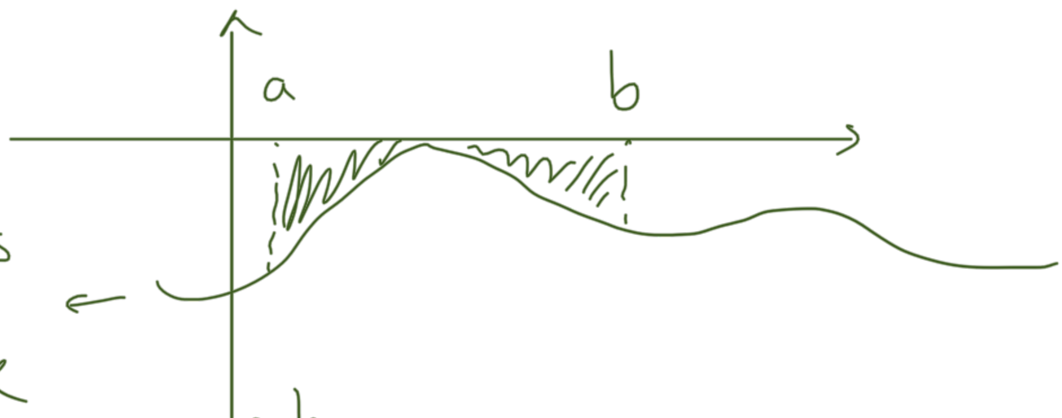
It is easy to see that for any partition P of $[-1,1]$, $U(f,P)=0$ and therefore $\inf\limits_{\text{over all } P} U(f,P) = 0$ It remains to show that $\sup\limits_{\forall P} L(f,P) = 0$.

for this notice that $L(f, P) = -\varepsilon$ where ε is the length of the interval around 0 in partition P. therefore by letting $\varepsilon \to 0$ we see that $\sup_{\forall P} L(f, P) = 0$.

$\Rightarrow f \in \mathcal{R}$ over $[-1, 1]$ and $\int_{-1}^{1} f = 0$

If your function "has jumps" in a finite number of points, the same result holds. (with the same proof)

for 3(b)



note that g is always negative

do you see that $\int_a^b g$ can not be positive?

$*$ Also you have in your lecture notes

$$g_1 \leq g_2 \text{ on } [a,b] \Rightarrow \int_a^b g_1 \leq \int_a^b g_2 \quad (*)$$

To show that $g = 0$, assume that $g \neq 0$ therefore $\exists \alpha \in [-1, 1]$ s.t. $g(\alpha) < 0$ since g is continuous there has to be a neighbourhood $(\alpha - \delta, \alpha + \delta)$ on which $g(x) < 0$. Use this neighbourhood to show that $\int_{-1}^{1} g < 0$ which is a contradiction.

solution (b)

constant function 0.

from (*) since $g \leq 0 \implies \int_{-1}^{1} g \leq \int_{-1}^{1} 0 = 0$

but the assumption says $\int_{-1} g \geq 0$

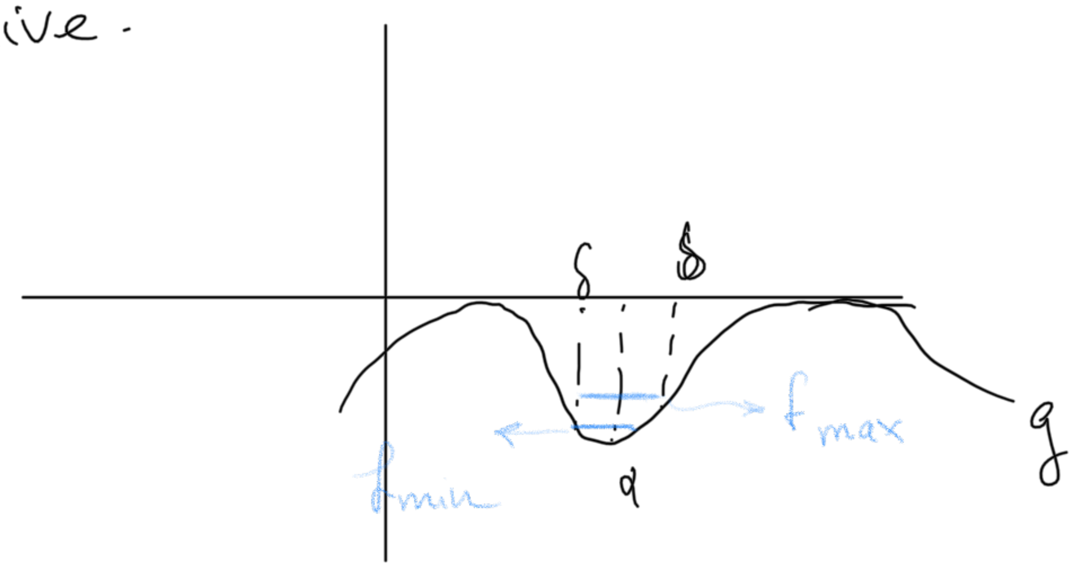
There fore, we have $\int_{-1}^{1} g = 0$.

Therefore you can see that for any partitions we have $L(g, P) \leq L(g, P') = \sum_{i=1}^{n} r_i$

where P' is $P \cup \{\alpha - \delta, \alpha + \delta\}$ and $r_i \leq 0 \ \forall i$

and the sum of the boxes between $\alpha - \delta$ and $\alpha + \delta$

is a negative number between $2\delta \times f_{min}$ and $2\delta \times f_{max}$ where f_{min} and f_{max} are the min and the max of f in $[\alpha - \delta, \alpha + \delta]$. and both are strictly negative.



Therefore, there is no x s.t. $g(x) < 0 \implies g = 0$

paoblem 4.

- You might have heard that rational numbers $\mathbb{Q}$ are dense in $\mathbb{R}$. This means any interval $[a,b]$ that you choose in $\mathbb{R}$ (no matter how small) it will contain some points from $\mathbb{Q}$.

- Check the values $U(f,P)$ and $L(f,P)$ for all partitions P of $[0,1]$.

Just for fun: Let $f(x) := \begin{cases} -1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$

if $f^2 \in \mathcal{R}$ over $[0,1]$?

solution: Observe that $\mathbb{Q}$ is dense in $\mathbb{R}$.

and $U(f,P) = 1 \Rightarrow \inf U(f,P) = 1 \quad \forall P$

and $L(f,P) = -1 \Rightarrow \sup L(f,P) = -1 \quad \forall P$

Because in any box, no mater how small, there always is $f = 1$ and $f = -1$ so the upper boxes and lower boxes are

are always the multiplicaions of their intervals with 1 and -1, respectively.

f^2 is the constant function $f^2 = 1 \Rightarrow f^2 \in \mathcal{R}$.

problem 5.

* Can you find some sequence where the sum of the first n elements is finite for any n but if $n \to \infty$ then the sum goes to ∞?

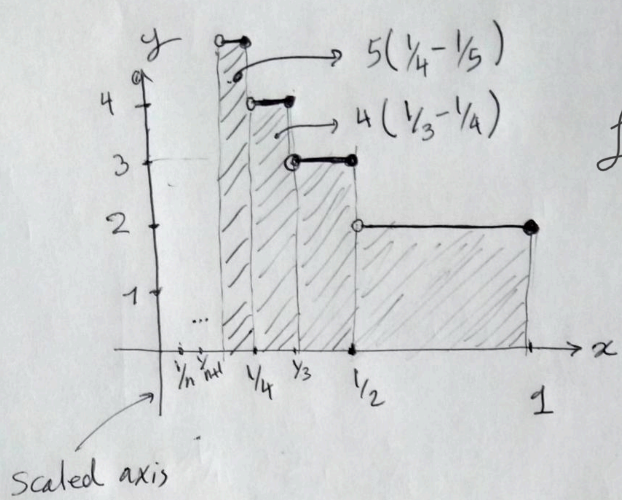
* Note that if f is unbounded then it is not integrable.

* Remember Ex 3.

(c) Let $g : [0,1] \to \mathbb{R}$, $g(x) = \int_x^1 f$

Can you see that the problem is equivalent to showing whether g is continuous at 0 or not? (from right)

5. a) let us consider the function f as



$$f(x) = \begin{cases} n+1 & : x \in \left(\frac{1}{n+1} ; \frac{1}{n}\right] \quad n \in \mathbb{N} \\ 0 & , \ x = 0 \end{cases}$$

then one can write the integral of f

$$\int_{1/n}^{1} f = n\left(\frac{1}{n-1} - \frac{1}{n}\right) + n-1\left(\frac{1}{n-2} - \frac{1}{n-1}\right) + \cdots + 2\left(\frac{1}{2}\right)$$

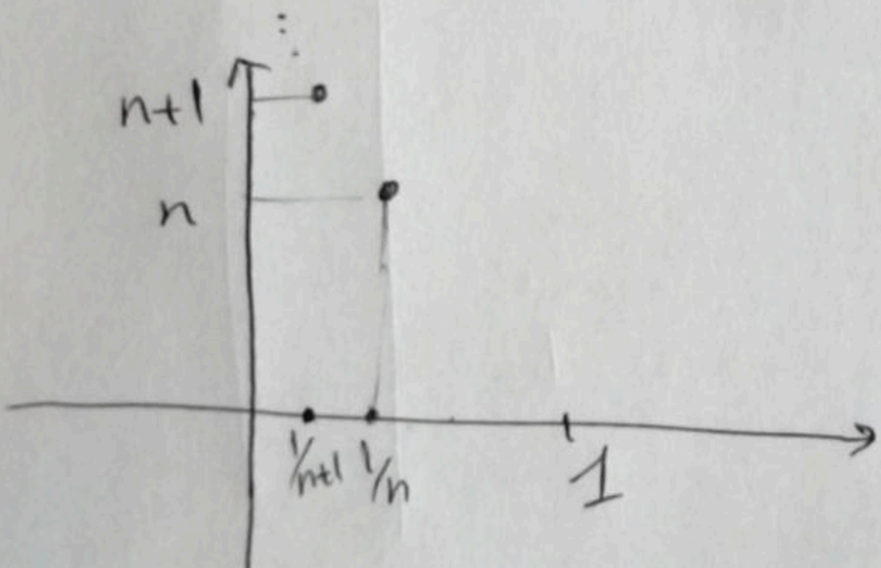
$$= \frac{1}{n-1} + \frac{1}{n-2} + \cdots + \frac{1}{2} + 1$$

Simplify

and $\lim\limits_{n \to \infty} \int_{1/n}^{1} f = \lim\limits_{n \to \infty} \left(\sum_{i=1}^{n-1} \frac{1}{i} \right)$

this is the Harmonic Series and it diverges.

b) let $f(x) := \begin{cases} n & \text{if } x = 1/n \\ 0 & \text{otherwise} \end{cases}$



then $\int_{1/n}^{1} f = 0$

and $\lim_{n \to \infty} \int_{1/n}^{1} f = 0$

However f is unbounded on $[0,1]$ and therefore not integrable.

c) the assumption is $f \in R([0,1])$

i.e f is a Riemann-integrable function on $[0,1]$

let us $\frac{1}{n} = x$; therefore we have to

Show :
$$\lim_{x \to 0^+} \int_x^1 f = \int_0^1 f$$

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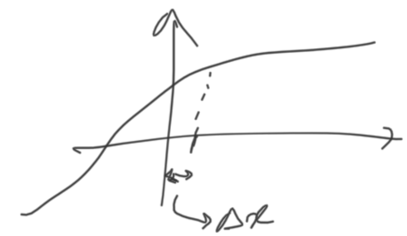
Since f is integrable on whole $[0,1]$, it is also true $\forall x \in [0,1]$ $f \in R([x,1])$.

therefore, one can define $g(x) := \int_x^1 f$

which is a real-valued function. $(g : [0,1] \to \mathbb{R})$

Hence, the problem is reduced to showing:

$$\lim_{x \to 0^+} g(x) \overset{?}{=} g(0).$$



$$\lim_{x \to 0^+} g(x) \overset{?}{=} g(0) \iff \lim_{x \to 0^+} \left(g(x) - g(0) \right) \overset{?}{=} 0 \overset{\text{definition of } g}{\iff}$$

$$\lim_{x \to 0^+} \left(\int_x^1 f - \int_0^1 f \right) \overset{?}{=} 0 \underset{\text{problem 1}}{\iff} \lim_{x \to 0^+} \int_x^1 f - \left(\int_0^x f + \int_x^1 f \right) \overset{?}{=} 0$$

$$\iff \lim_{x \to 0^+} - \int_0^x f \overset{?}{=} 0 \overset{\times(-1)}{\iff} \lim_{x \to 0^+} \int_0^x f \overset{?}{=} 0$$

Now we use boundedness of f on $[0,x]$ to show limit is zero.

$$\lim_{x \to 0^+} \int_0^x f = \cancel{\lim_{x \to 0^+} \sup_{\forall P} L(f,P) = \lim_{x \to 0^+} \sup_{\forall P} \sum_{i=1}^{n} \inf f \, \Delta x_i}$$

~~f is bounded $\Rightarrow$ $\inf f$ is finite and therefore $\inf f \, \Delta x_i \xrightarrow{x \to 0^+} 0$~~

This is not wrong but I replaced it.

$$= \lim_{\Delta x \to 0} \sup_{\substack{\forall P \\ \text{partitions} \\ \text{of } \Delta x}} L(f,P)$$

Note that for any partition P of Δx

$$m \leq L(f,P) \leq M$$

$$\Rightarrow m \leq \sup_{\forall P} L(f,P) \leq M$$

$$\downarrow \qquad \Downarrow \qquad \downarrow$$

$$0 \qquad\qquad 0$$

$$\lim_{\Delta x \to 0} \sup_{\forall P} L(f,P) = 0 \checkmark$$

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