Analysis 2 / Leipzig University / SS 2020 / Problem sheet 9.

Problem 2. first make the RREF of the given SOLE:

$$\begin{array}{cc} & \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \end{array} \\ \begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \end{array} & \left[\begin{array}{cccc|c} 1 & 2 & 2 & 1 & 3 \\ 3 & 8 & 8 & 7 & 9 \\ 2 & 5 & 6 & 5 & 7 \\ 1 & 3 & 4 & r & 5 \end{array}\right] \end{array} \xrightarrow[r_4 = r_4 - r_1]{r_2 = r_2 - 3r_1,\ r_3 = r_3 - 2r_1} \begin{array}{cc} & \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \end{array} \\ \begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \end{array} & \left[\begin{array}{cccc|c} 1 & 2 & 2 & 1 & 3 \\ 0 & 2 & 2 & 4 & 0 \\ 0 & 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & r-1 & 2 \end{array}\right] \end{array}$$

switch r_2 and r_3

$$\begin{array}{cc} & \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \end{array} \\ \begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \end{array} & \left[\begin{array}{cccc|c} 1 & 2 & 2 & 1 & 3 \\ 0 & 1 & 2 & 3 & 1 \\ 0 & 2 & 2 & 4 & 0 \\ 0 & 1 & 2 & r-1 & 2 \end{array}\right] \end{array} \xrightarrow[r_4 = r_4 - r_2]{r_1 = r_1 - 2r_2,\ r_3 = r_3 - 2r_2} \begin{array}{cc} & \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \end{array} \\ \begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \end{array} & \left[\begin{array}{cccc|c} 1 & 0 & -2 & -5 & 1 \\ 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & 0 & r-4 & 1 \end{array}\right] \end{array}$$

$$\xrightarrow{r_3 = r_3 / -2} \begin{array}{cc} & \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \end{array} \\ \begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \end{array} & \left[\begin{array}{cccc|c} 1 & 0 & -2 & -5 & 1 \\ 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & r-4 & 1 \end{array}\right] \end{array} \xrightarrow[r_2 = r_2 - 2r_3]{r_1 = r_1 + 2r_3} \begin{array}{cc} & \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \end{array} \\ \begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \end{array} & \left[\begin{array}{cccc|c} 1 & 0 & 0 & -3 & 3 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & r-4 & 1 \end{array}\right] \end{array}$$

$$r_4 = \frac{r_4}{r-4}$$

(note that $r \neq 4$ because if $r = 4$ then $r - 4 = 0$ which means $0 \times x_4 = 1$ (from the last row of the matrix) and $0 = 1$ ↯)

$$\begin{array}{cc} & \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \end{array} \\ \begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \end{array} & \left[\begin{array}{cccc|c} 1 & 0 & 0 & -3 & 3 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1/(r-4) \end{array}\right] \end{array} \xrightarrow[r_2 = r_2 - r_4,\ r_3 = r_3 - r_4]{r_1 = r_1 + 3r_4} \begin{array}{cc} & \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \end{array} \\ \begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \end{array} & \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 + \frac{3}{r-4} \\ 0 & 1 & 0 & 0 & -1 - \frac{1}{r-4} \\ 0 & 0 & 1 & 0 & 1 - \frac{1}{r-4} \\ 0 & 0 & 0 & 1 & \frac{1}{r-4} \end{array}\right] \end{array}$$

For $r \in \mathbb{R} \setminus \{4\}$ the system is solvable with the solution set:

$$\left\{ \left(3 + \frac{3}{r-4}, -1 - \frac{1}{r-4}, 1 - \frac{1}{r-4}, \frac{1}{r-4}\right) \in \mathbb{R}^4 : r \neq 4 \right\}$$

(for example set $r=3$ then the point $(0,0,2,-1)$ is a solution, can you simply see it?)

Problem 4.

Since a matrix is invertibe if and only if the determinant is nonzero and the determinant of a triangular matrix is the multiplication of the elements on the diagonal, we get :

$$\boxed{A^{-1} \text{ exists} \iff A_{ii} \neq 0 \text{ for all } i}$$

Therefore assume that A is an $n \times n$ triangular matrix and for all $i \in \{1, \dots, n\}$, $A_{ii} \neq 0$.

Let B be the inverse of A. i.e., $AB = BA = \underset{\text{identity matrix.}}{I_n}$

Let A be a lower triangular matrix. $A = \begin{bmatrix} a_{11} & & 0 \\ a_{21} & a_{22} & \\ \vdots & & \ddots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$

we want to show that $\underbrace{b_{ij} = 0 \quad \forall j > i}$ which means B is also lower triangular. Moreover $b_{ii} = \frac{1}{a_{ii}}$ $\forall i \in \{1, \dots, n\}$. We use the equality $AB = I_n$:

$$\begin{array}{c} r_1 \to \\ r_2 \to \\ \vdots \\ r_n \to \end{array}\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0\cdots & 0 \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{bmatrix} \begin{array}{c} \begin{array}{cccc} c_1\downarrow & c_2\downarrow & \quad & c_n\downarrow \end{array} \\ \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}\end{array} = \begin{bmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{bmatrix}$$

let r_i be the i-th row of A and c_j be the j-th column of the B.

from the matrix equality above we see that:

$$1 = r_1 c_1 = a_{11}b_{11} + 0\cdot\cancelto{0}{b_{12}} + 0\cdot\cancelto{0}{b_{13}} + \cdots + 0\cdot\cancelto{0}{b_{1n}} = a_{11}b_{11}$$

$$\Rightarrow b_{11} = \frac{1}{a_{11}} \quad \Big/ \quad \text{and} \quad 0 = r_1 c_j = a_{11}b_{1j} + \cancelto{0}{0\cdot b_{2j} + \cdots + 0\cdot b_{nj}} \quad \forall j \geq 2$$

$$\overset{(a_{11}\neq 0)}{\Longrightarrow} b_{1j} = 0 \quad \forall j \geq 2$$

Now we can rewrite the equality again:

$$\begin{array}{c} r_1 \to \\ r_2 \to \\ \vdots \\ r_n \to \end{array}\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0\cdots & 0 \\ \vdots & & 0 & \vdots \\ a_{n1} & \cdots & & a_{nn}\end{bmatrix} \begin{array}{c} \begin{array}{cccc} c_1\downarrow & c_2\downarrow & \quad & c_n\downarrow \end{array} \\ \begin{bmatrix} b_{11} & 0 & \cdots & 0 \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}\end{array} = \begin{bmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{bmatrix}$$

from $1 = r_2 \cdot c_2 = \cancelto{0}{a_{21} \times 0} + a_{22} \cdot b_{22} + 0 \cdot \cancelto{0}{b_{23}} + \cdots + 0 \cdot \cancelto{0}{b_{2n}} = a_{22}b_{22}$

we get $b_{22} = \frac{1}{a_{22}}$, and

$$0 = r_2 c_j = \cancelto{0}{a_{21}} \cdot 0 + a_{22} \cdot b_{2j} + \cancelto{0}{0 \cdot b_{3j} + \cdots + 0 \cdot b_{nj}}$$

$(a_{22} \neq 0)$

$$\Rightarrow \quad b_{2j} = 0 \quad \forall_j \geq 3.$$

rewriting the equality again :

$$\begin{bmatrix} a_{11} & & & \\ a_{21} & a_{22} & & 0 \\ \vdots & & \ddots & \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & 0 & \cdots & 0 \\ b_{21} & b_{22} & 0 & \cdots 0 \\ & & \ddots & \\ b_{n1} & \cdots & & b_{nn} \end{bmatrix}$$

Assume that $i \geq 2$ and

$$\forall_k < i \quad : \quad b_{kk} = \frac{1}{a_{kk}} \quad \text{and} \quad b_{kj} = 0 \quad \forall_{j>k}.$$

$r_i \rightarrow$, $c_j \downarrow$

$$\begin{bmatrix} a_{11} & & & & & \\ \vdots & a_{22} & & & 0 & \\ \vdots & & \ddots & & & \\ & & & a_{i-1\,i-1} & & \\ \vdots & & & & a_{ii} & \\ & & & & & \ddots \\ a_{n1} & \cdots & & & & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & & & & & \\ & b_{22} & & & 0 & \\ \vdots & & \ddots & & & \\ & & & b_{i-1\,i-1} & 0 \cdots & 0 \\ \vdots & & & b_{i2} \cdots & b_j \cdots & b_{in} \\ & & & & \ddots & \vdots \\ b_{n1} & \cdots & & & & b_{nn} \end{bmatrix} = I_n$$

$$\text{from } 1 = r_i c_i = a_{i1} \times 0 + \cancelto{0}{a_{i\,i-1}} \times 0 + a_{ii} \cdot b_{ii} + \cancelto{0}{0 \cdot b_{i+1\,i}} + \cdots + 0 \cdot b_{ni}$$

$$\Rightarrow b_{ii} = \frac{1}{a_{ii}}, \quad \text{and} \quad \forall_{j>i} :$$

$$0 = r_i C_j = \underbrace{a_{i1} \cdot 0 + \cdots + a_{i\,i-1} \times 0}_{0} + a_{ii} b_{ij} + \underbrace{0 \cdot b_{i+1\,j} + \cdots + 0 \times b_{nj}}_{0}$$

$(a_{ii} \neq 0)$

$$\Longrightarrow \quad b_{ij} = 0$$

Therefore the proof is completed and B is also lower triangular with $B_{ii} = \dfrac{1}{A_{ii}}$.

The proof for upper triangular is similar. ▦

Problem 6.

we want C_1, C_2, C_3 to be lin. indep. or in another words, we do not want any of them to be a linear combination of the other two.

we can check this

$$\begin{array}{c} C_2 \\ C_3 \\ C_1 \end{array} \begin{bmatrix} 1 & 3 & 2 & 1 \\ 2 & 2 & 4 & t \\ 5 & 7 & s & 2 \end{bmatrix} \xrightarrow[C_1 = C_1 - 5C_2]{C_3 = C_3 - 2C_2} \begin{array}{c} C_2 \\ C_3 \\ C_1 \end{array} \begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & -4 & 0 & t-2 \\ 0 & -8 & s-10 & -3 \end{bmatrix}$$

$$\xrightarrow{C_3 = \frac{C_3}{-4}} \begin{array}{c} C_2 \\ C_3 \\ C_1 \end{array} \begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & 1 & 0 & \frac{2-t}{4} \\ 0 & -8 & s-10 & -3 \end{bmatrix} \xrightarrow[C_1 = C_1 + 8C_3]{C_2 = C_2 - 3C_3} \begin{array}{c} C_2 \\ C_3 \\ C_1 \end{array} \begin{bmatrix} 1 & 0 & 2 & 1-\frac{3}{4}(2-t) \\ 0 & 1 & 0 & \frac{2-t}{4} \\ 0 & 0 & s-10 & -3+2(2-t) \end{bmatrix}$$

if $s \neq 10$ then divide the last row by $s-10$ and this gives $\underline{3}$ lin. indep. rows.

if $S = 10$ then we need to have

$-3+2(2-t) = -3+4-2t = 1-2t \neq 0 \Rightarrow$

$t \neq \frac{1}{2} \Rightarrow$

for $(s,t) \in \mathbb{R}^2 \setminus \{(b, \frac{1}{2})\}$

c_1, c_2 and c_3 are linearly indep. in $\mathbb{R}^4$.

Problem 8.

The definitions:

$$\varepsilon_{ijk} = \begin{cases} +1 & \rightsquigarrow \quad (\text{even permutation}) \\ -1 & \rightsquigarrow \quad (\text{odd permutation}) \\ 0 & \text{else} \end{cases}$$

$$\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

you can simply check that $\varepsilon_{ijk} = \det \begin{bmatrix} e_i & e_j & e_k \end{bmatrix}_{3\times 3}$

where e_n is a column vector with the n-th element equal to $\underline{1}$ and the rest zero.

(easy example: $\varepsilon_{123} = \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1$)

(i) we want to prove $\sum_{i=1}^{3} \varepsilon_{ijk} \varepsilon_{ipq} = \delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}$

if $j=k$ then both sides are zero.

The left hand side always have $\varepsilon_{ijk}=0$ for all $i\in\{1,2,3\}$

and the RHS $\delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp} \overset{j=k}{=} \delta_{kp}\delta_{kq} - \delta_{kq}\delta_{kp} = 0$

Now let $j \neq k$, there exists an $i \in \{1,2,3\}$ s.t.

i, j, k are all different and

$$\varepsilon_{ijk}\varepsilon_{ipq} = \det\begin{pmatrix} | & | & | \\ e_i & e_j & e_k \\ | & | & | \end{pmatrix} \cdot \det\begin{pmatrix} | & | & | \\ e_i & e_p & e_q \\ | & | & | \end{pmatrix}$$

$$\overset{\det(A)=\det(A^T)}{=} \det\begin{pmatrix} | & | & | \\ e_i & e_j & e_k \\ | & | & | \end{pmatrix} \cdot \det\begin{pmatrix} - & e_i & - \\ - & e_p & - \\ - & e_q & - \end{pmatrix}$$

$\det A \cdot \det B = \det(AB) = \det(BA)$

$$= \det\left[\begin{pmatrix} - & e_i & - \\ - & e_p & - \\ - & e_q & - \end{pmatrix} \times \begin{pmatrix} | & | & | \\ e_i & e_j & e_k \\ | & | & | \end{pmatrix}\right]$$

$$= \det\begin{bmatrix} e_i^T e_i & e_i^T e_j & e_i^T e_k \\ e_p^T e_i & e_p^T e_j & e_p^T e_k \\ e_q^T e_i & e_q^T e_j & e_q^T e_k \end{bmatrix} = \det \begin{array}{c} \begin{array}{ccc} & i\neq j & i\neq k \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ \delta_{pi} & \delta_{pj} & \delta_{pk} \\ \delta_{qi} & \delta_{qj} & \delta_{qk} \end{bmatrix} \end{array}$$

$$= 1 \times \det \begin{pmatrix} \delta_{pj} & \delta_{pk} \\ \delta_{qj} & \delta_{qk} \end{pmatrix} = \delta_{pj}\delta_{qk} - \delta_{pk}\delta_{qj}$$

$$= \delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp} \cdot (*)$$

Note that if j, i, k are not different then $i=j$ or $i=k$ and both make $\varepsilon_{ijk} = 0$ therefore

for the unique i that makes $\{i, j, k\}$ all different.

$$\sum_{i=1}^{3} \varepsilon_{ijk}\varepsilon_{ipq} = \varepsilon_{ijk}\varepsilon_{ipq} \overset{(*)}{=} \delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}.$$

$\square$

ii) WTS $u \times (v \times w) = (u \cdot w)v - (u \cdot v)w$

NOTE: $u \cdot v = u_i \delta_{ij} v_j$

and $u \times v = u_i v_j \varepsilon_{ijk} e_k$ (2)

however we do not write the summation sign ($\sum$) you need to know that these are sums over all i, j, k.

$$u \times (v \times w) \overset{(2)}{=} u_i \overbrace{(v \times w)_j} \varepsilon_{ijk} e_k \overset{(2)}{=}$$

$$= u_i \left(v_m w_n \overbrace{\varepsilon_{mnj}}^{=\varepsilon_{jmn}} \right) \overbrace{\varepsilon_{ijk}}^{=\varepsilon_{jki}} e_k =$$

$$u_i v_m w_n \left(\varepsilon_{jmn} \varepsilon_{jki} \right) e_k$$

$$\overset{(2)}{=} u_i v_m w_n \left(\delta_{mk} \delta_{ni} - \delta_{mi} \delta_{nk} \right) e_k$$

$$= u_i v_m w_n \left(\delta_{mk} \delta_{ni} \right) e_k - u_i v_m w_n \left(\delta_{mi} \delta_{nk} \right) e_k$$

$$= \left(u_i \delta_{ni} w_n \right) v_m \delta_{mk} e_k - \left(u_i \delta_{mi} v_m \right) w_n \delta_{nk} e_k$$

$$= (u \cdot w) v_k e_k - (u \cdot v) w_k e_k$$

$$= (u \cdot w) v - (u \cdot v) w . \quad \blacksquare$$

iii) WTS $u\times(v\times w)+v\times(w\times u)+w\times(u\times v)=0$

NOTE That $a\cdot b=b\cdot a$ for all vectors $a,b\in\mathbb{R}^3$.

we use (ii)

$$u\times(v\times w)=\cancel{(u\cdot w)}v-\cancel{(u\cdot v)}w$$

$$v\times(w\times u)=\cancel{(v\cdot u)}w-\cancel{(v\cdot w)}u$$

$$w\times(u\times v)=\cancel{(w\cdot v)}u-\cancel{(w\cdot u)}v$$

by adding up the right hand sides of all the three equations you see the cancellation showed by colors. Therefore, they sum up to zero.

$\blacksquare$

Problem 1: Form the augmented matrix $A' = (A \mid b)$

then reduce A' to RREF using row ops.

$$A' \sim \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -8 & 9 \\ 0 & 1 & 0 & 0 & 11/4 & -1 \\ 0 & 0 & 1 & 0 & 1/2 & 1 \\ 0 & 0 & 0 & 1 & -3/4 & 0 \end{array}\right)$$

Since the two systems are equivalent, they have the same solutions. Therefore, we have the following system:

$$\begin{cases} x_1 - 8x_5 = 9 \\ x_2 + \frac{11}{4}x_5 = -1 \\ x_3 + \frac{1}{2}x_5 = 1 \\ x_4 - \frac{3}{4}x_5 = 0 \end{cases}$$

Taking $x_5 \in \mathbb{R}$ as arbitrary, we arrive at the following solution set.

$$X = \left\{ (x_1, x_2, x_3, x_4, x_5) = (9 + x_5, -1 - 11/4\, x_5, 1 - 1/2\, x_5, 3/4\, x_5, x_5) : x_5 \in \mathbb{R} \right\}$$

problem 3:

i) $\det(A) = 149$

ii) $\det(B) = -6$

iii) $(A^5)_{4\times4}$ and $(B^2)_{5\times5}$ therefore their product is not defined.

iv) using induction on $n \equiv$ matrix length we have:

1) Base Case: $n=2$

$$\det(V) = \begin{vmatrix} 1 & 1 \\ a_1 & a_2 \end{vmatrix} = a_2 - a_1$$

2) Inductive Case: Assume the formula holds for $n\times n$ matrix and show it holds for $(n+1)\times(n+1)$ matrix.

$$V = \begin{array}{c} r_1 \\ r_2 \\ r_3 \\ \vdots \\ r_n \\ r_{n+1} \end{array} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_{n+1} \\ a_1^2 & a_2^2 & \cdots & a_{n+1}^2 \\ \vdots & \vdots & & \vdots \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_{n+1}^{n-1} \\ a_1^n & a_2^n & \cdots & a_{n+1}^n \end{bmatrix}_{(n+1)\times(n+1)}$$

$$r_{n+1} \leftarrow r_{n+1} - a_1 r_n$$

$$\Rightarrow \det V = \begin{array}{c} \\ \\ \\ \\ \\ r_n \\ r_{n+1} \end{array} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_{n+1} \\ a_1^2 & a_2^2 & \cdots & \vdots \\ \vdots & \vdots & & \\ a_1^{n-1} & a_2^{n-1} & & \\ 0 & a_2^n - a_1 a_2^{n-1} & \cdots & a_{n+1}^n - a_1 a_{n+1}^{n-1} \end{vmatrix}$$

similarly $r_n \leftarrow r_n - a_1 r_{n-1}$.

Doing this all the way up to r_2 step-by-step results in the following determinant:

$$\left| V_{(n+1)\times(n+1)} \right| = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & a_2 - a_1 & \cdots & a_{n+1} - a_1 \\ 0 & a_2^2 - a_1 a_2 & \cdots & \vdots \\ \vdots & \vdots & & \\ 0 & a_2^n - a_1 a_2^{n-1} & \cdots & a_{n+1}^n - a_1 a_{n+1}^{n-1} \end{vmatrix}$$

$$\overset{\text{expand by } c_1}{=} 1. \begin{vmatrix} a_2 - a_1 & a_3 - a_1 & \cdots & a_{n+1} - a_1 \\ a_2^2 - a_1 a_2 & a_3^2 - a_1 a_3 & \cdots & a_{n+1}^2 - a_1 a_{n+1} \\ \vdots & \vdots & & \vdots \\ a_2^n - a_1 a_2^{n-1} & a_3^n - a_1 a_3^{n-1} & \cdots & a_{n+1}^n - a_1 a_{n+1}^{n-1} \end{vmatrix}$$

$$\overset{\text{factor}}{=} \begin{vmatrix} 1.(a_2 - a_1) & 1.(a_3 - a_1) & \cdots & 1.(a_{n+1} - a_1) \\ a_2 \cdot (a_2 - a_1) & a_3(a_3 - a_1) & \cdots & a_{n+1}(a_{n+1} - a_1) \\ \vdots & \vdots & & \vdots \\ a_2^{n-1}(a_2 - a_1) & a_3^{n-1}(a_3 - a_1) & \cdots & a_{n+1}^{n-1}(a_{n+1} - a_1) \end{vmatrix}$$

Now we can factor out from each column one factor and determinant is respectively multiplied by that factor.

$$\Rightarrow \det\left(V_{(n+1)(n+1)}\right) = \prod_{1<j\leq n+1} (a_j - a_1) \cdot \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_2 & a_3 & & a_{n+1} \\ \vdots & & & \\ a_2^{n-1} & a_3^{n-1} & & a_{n+1}^{n-1} \end{vmatrix}$$

Now we have det. of an $n \times n$ matrix of the same form. By Hypothesis of the Inductive Step we have

$$\det\begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_2 & a_3 & & a_{n+1} \\ \vdots & \vdots & & \\ a_2^{n-1} & a_3^{n-1} & & a_{n+1}^{n-1} \end{pmatrix} = \prod_{2\leq i<j\leq n+1} (a_j - a_i)$$

$$\Rightarrow \det(V) = \prod_{1<j\leq n+1} (a_j - a_1) \cdot \prod_{2\leq i<j\leq n+1} (a_j - a_i)$$

$$= \prod_{1\leq i<j\leq n+1} (a_j - a_i)$$

which is the formula for $(n+1)\times(n+1)$ matrix. Therefore, by principle of induction the formula is correct $\forall n \geq 2$.

Problem 5 :

$$A_1^{-1} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ -2 & -2 & 1 & 1 \\ 2 & 3 & -1 & -1 \\ 2 & 0 & -1 & 0 \end{bmatrix} , \quad A_2^{-1} = \frac{1}{17}\begin{bmatrix} -9 & -12 & 1 & 17 \\ 2 & -3 & -4 & 0 \\ 5 & 1 & 7 & 0 \\ 15 & 20 & 4 & -17 \end{bmatrix}$$

for b_2 we get only Trivial Solution.

$$A_1 X = b_2 \xRightarrow{\cdot A_1^{-1}} X = 0 \quad ; \quad A_2 X = b_2 \Rightarrow X = 0$$

Problem 7 : $v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \quad v_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

$$u_1 = \frac{v_1}{\| v_i \|} \quad ; \quad \| v_1 \| = \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2}$$

$$\Rightarrow u_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$u_2 = \frac{v_2 - \langle v_2 , u_1 \rangle u_1 \quad := k_1}{\| k_1 \|}$$

$$\langle v_2 , u_1 \rangle = 0 \cdot 2 + \frac{1}{\sqrt{2}} \cdot 1 + \frac{1}{\sqrt{2}} \cdot 0 = \frac{1}{\sqrt{2}}$$

$$\Rightarrow k_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1/2 \\ -1/2 \end{pmatrix}$$

$$\Rightarrow \| k_1 \| = \sqrt{2^2 + \left(\tfrac{1}{2}\right)^2 + \left(-\tfrac{1}{2}\right)^2} = \sqrt{4 + 1/4 + 1/4} = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$$

$$\Rightarrow u_2 = \frac{\sqrt{2}}{3} \begin{pmatrix} 2 \\ 1/2 \\ -1/2 \end{pmatrix} = \frac{\sqrt{2}}{6} \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix}$$

And $$u_3 = \frac{v_3 - \langle v_3 , u_2 \rangle u_2 - \langle v_3 , u_1 \rangle u_1 \quad := k_2}{\| k_2 \|}$$

$$\langle v_3, u_2 \rangle = 2 \cdot \frac{\sqrt{2}}{6} \cdot 4 - \frac{\sqrt{2}}{6} = 7 \frac{\sqrt{2}}{6}$$

$$\langle v_3, u_1 \rangle = 1/\sqrt{2}$$

$$\Rightarrow k_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} - 7\frac{\sqrt{2}}{6} \cdot \frac{\sqrt{2}}{6} \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix} - \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} - \frac{7}{18} \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 1/3 \\ -2/3 \\ 2/3 \end{pmatrix}$$