Problem Sheet 12

Due Date: 08.06.2020, 12:00 UTC+2 (CEST)

Problem 1. [**4** pts] Let $[a,b] \subset \mathbb{R}$, let $x_0 \in \mathbb{R}$ be such that $x_0 \in [a,b]$. Show that the following holds

$$f \in \mathcal{R} \text{ over } [a,b] \iff \begin{cases} f \in \mathcal{R} \text{ over } [a,x_0] \\ f \in \mathcal{R} \text{ over } [x_0,b] \end{cases}$$

and then $\int_a^b f = \int_a^{x_0} f + \int_{x_0}^b f$.

Problem 2. [**1+2** pts] Let $f \in \mathcal{R}$. Show that: (a) $sin f$, (b) $min(f,0)$ are in $\mathcal{R}$.

Problem 3. [**2+2** pts]

(a) Let

$$f(x) = \begin{cases} -1 & \text{for } x = 0 \\ 0 & \text{for } x \neq 0 \end{cases}$$

Show that $f \in \mathcal{R}$ and that $\int_{-1}^1 f = 0$.

(b) Let $g : [-1,1] \to (-\infty, 0]$ be continuous. Show that

$$\int_{-1}^1 g \geq 0 \implies \forall_{x \in [-1,1]}\, g(x) = 0$$

Why does the result for g does not hold for f?

Problem 4. [**3** pts] Consider

$$f(x) = \begin{cases} 1 & \text{for } x \in \mathbb{Q} \\ -1 & \text{for } x \notin \mathbb{Q} \end{cases}$$

does f belong to $\mathcal{R}$ over $[0,1]$? Does f^2 belong to $\mathcal{R}$ over $[0,1]$?

Problem 5. [**4** pts] Suppose $f \in \mathcal{R}$ over $[1/n, 1]$ for any $n \in \mathbb{N}$
(a) does $\lim_{n\to\infty} \int_{\frac{1}{n}}^1 f$ always exist?
(b) if $\lim_{n\to\infty} \int_{\frac{1}{n}}^1 f$ exists, does it mean that $f \in \mathcal{R}$ over $[0,1]$?
(c) if $f \in \mathcal{R}$ over $[0,1]$, do we have $\lim_{n\to\infty} \int_{\frac{1}{n}}^1 f = \int_0^1 f$?