HW11. Review + Hints

problem 1 is similar to Ex1:

Ex 1. For the following linear map φ find basis and dimension of $\ker\varphi$ and $\operatorname{im}\varphi$.

$$\varphi : \mathbb{R}^5 \longrightarrow \mathbb{R}^4$$

$$(x_1, x_2, x_3, x_4, x_5) \longmapsto (3x_1 + 9x_2 - 2x_3 + 17x_4 - 13x_5,$$
$$2x_1 + 7x_2 + 7x_4 - 2x_5,$$
$$2x_1 + 5x_2 - 2x_3 + 13x_4 - 13x_5,$$
$$x_1 + 3x_2 - x_3 + 5x_4 - 4x_5)$$

[hint: use HW9, problem 1.]

Solution:

First we find $\ker\varphi$. By definition:

$$\ker\varphi = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : \varphi(x_1, \dots, x_5) = 0\}$$

This means we are looking for $X = (x_1, \dots, x_5) \in \mathbb{R}^5$ such that

$Ax = 0$ where:

$$A = \begin{bmatrix} 3 & 9 & -2 & 17 & -13 \\ 2 & 7 & 0 & 7 & -2 \\ 2 & 5 & -2 & 13 & -13 \\ 1 & 3 & -1 & 5 & -4 \end{bmatrix}$$

from HW9 - Problem 1, RREF of $(A|0)$ is

$$B = \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -8 & 0 \\ 0 & 1 & 0 & 0 & 11/4 & 0 \\ 0 & 0 & 1 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 & -3/4 & 0 \end{array}\right]$$

By writing down the simplified system we get:

$$\begin{cases} x_1 - 8x_5 = 0 \\ x_2 + \frac{11}{4} x_5 = 0 \\ x_3 + \frac{1}{2} x_5 = 0 \\ x_4 - \frac{3}{4} x_5 = 0 \end{cases}$$

$\Rightarrow$ the solution space is $\ker \varphi = X \subset \mathbb{R}^5$:

$$X = \left\{ \left(8t, \frac{-11}{4}t, \frac{-1}{2}t, \frac{3}{4}t, t\right) ; t \in \mathbb{R} \right\}$$

Therefore, $\ker \varphi$ is a line in $\mathbb{R}^5$ spanned by the vector $v = \left(8, \frac{-11}{4}, \frac{-1}{2}, \frac{3}{4}, 1\right)$ and $\dim X = 1$.

From Lemma 2.1 (Lecture notes):

$$\dim \mathbb{R}^5 = \dim \ker \varphi + \dim \operatorname{im} \varphi$$

$$\Rightarrow 5 = 1 + \dim \operatorname{im} \varphi$$

$$\Rightarrow \dim \operatorname{im} \varphi = 4$$

To find a basis for $\operatorname{im} \varphi$, all you need to do is finding four vectors $v_1, v_2, v_3, v_4 \in \mathbb{R}^5$ such that the five vectors v_1, v_2, v_3, v_4, v form a basis for $\mathbb{R}^5$. In other words, we want v_1, v_2, v_3, v_4, v to be linearly independent. Once you find them then $\varphi(v_1), \varphi(v_2), \varphi(v_3), \varphi(v_4)$ form a basis for $\operatorname{im} \varphi$.

One fast way to find $v_1, \dots, v_4$ (which are not unique)

$(1,0,0,0,0)$ --- $(0,0,0,1,0)$

is to guess! For example just set $v_1 = e_1, \dots, v_4 = e_4$ then check whether v_1, v_2, v_3, v_4, v are linearly indep.

If yes, you are done. If no, then make small changes until they are lin. indep. Another way which is more formal is that first find v_1 by solving $v \cdot v_1 = 0$ then solve $\begin{cases} v \cdot v_2 = 0 \\ v_1 \cdot v_2 = 0 \end{cases}$ to find v_2 and so on.

In this case setting $v_i = e_i$ gives $v_1, \ldots, v_4, v$ lin. indep. Since

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 8 & -\frac{11}{4} & -\frac{1}{2} & \frac{3}{4} & 1 \end{bmatrix} \xrightarrow{\text{RREF}} I_5.$$

Thus, $\varphi(e_1), \varphi(e_2), \varphi(e_3), \varphi(e_4)$ is a basis for $\operatorname{im}\varphi$.

$$\varphi(e_1) = (3,2,2,1)$$
$$\varphi(e_2) = (9,7,5,3)$$
$$\varphi(e_3) = (-2,0,-2,-1)$$
$$\varphi(e_4) = (17,7,13,5).$$

$\blacksquare$

Problem $\underline{2}$ is similar to Ex 2 :

Ex 2. Find the eigenpairs (eigenvalues + corresponding eigenvectors) of the tensor:

$$\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$(x_1, x_2, x_3) \longmapsto (x_2 + 2x_3 , -4x_1 + x_2 + 4x_3 , -5x_1 + x_2 + 7x_3)$$

Solution:

$\lambda \in \mathbb{R}$ is an eigenvalue for φ if $\varphi(v) = \lambda v$

for some $v \overset{\neq 0}{=} (v_1, v_2, v_3) \in \mathbb{R}^3$. we call $\frac{v}{\|v\|}$ the
normalized
eigenvector corresponding to λ.

Let $A = \begin{bmatrix} 0 & 1 & 2 \\ -4 & 1 & 4 \\ -5 & 1 & 7 \end{bmatrix}$, then $\varphi(v) = Av$.

The eigenvalues of φ are obtained by solving

$$Av = \lambda v \Rightarrow (A - \lambda I)v = 0 \quad \text{since } v \neq 0$$

we should have $\det(A - \lambda I) = 0$

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 & 2 \\ -4 & 1-\lambda & 4 \\ -5 & 1 & 7-\lambda \end{bmatrix} = (\lambda - 1)(\lambda - 2)(\lambda - 5)$$

$$\Rightarrow \lambda_1 = 1 \quad \lambda_2 = 2 \quad \lambda_3 = 5.$$

To find the corresponding eigenvectors we do the following:

$$Av = \lambda_1 v \Rightarrow (A - \lambda_1 I)v = 0 \Rightarrow \begin{bmatrix} -1 & 1 & 2 \\ -4 & 0 & 4 \\ -5 & 1 & 6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

where $v = (v_1, v_2, v_3) \in \mathbb{R}^3 \setminus \{0\}$.

solving the SOLE

gives $\{(v_1, v_2, v_3) = (t, -t, t) : t \in \mathbb{R}\}$

a normalized vector on this solution space

is $v = \left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

Similarly solving $Av = \lambda_2 v$ and $Av = \lambda_3 v$

we get the pairs:

$$\left.\begin{aligned} \lambda_1 = 1 &\longrightarrow v = \left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \\ \lambda_2 = 2 &\longrightarrow u = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) \\ \lambda_3 = 5 &\longrightarrow w = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right) \end{aligned}\right\} \text{eigenpairs}$$

* Note that not always for an $n \times n$ matrix we have n distinct eigenvalues. and not always only one eigenvector corresponds to an eigenvalue.

Problem 3. Let $\{a^1, \dots, a^d\}$ be a basis of V. Let $S: V \to V$ be a tensor s.t.

$$Sa^i = a^{i+1} \text{ for } i = 1, \dots, d-1 \text{ and } Sa^d = \alpha_i a^i : \alpha_i \in \mathbb{R}, \ 1 \leq i \leq d$$

Find the characteristic polynomial of S.

Remember: $p(x) = \det(S - xI)$ where S is a matrix representation of the tensor S.

(1) find the characteristic polynomial of the tensor $\varphi: \mathbb{R}^d \to \mathbb{R}^d$ defined by the matrix

$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 & \alpha_1 \\ 1 & 0 & \cdots & 0 & \alpha_2 \\ 0 & 1 & \cdots & 0 & \alpha_3 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & \alpha_d \end{bmatrix}$$

(2). Argue that the matrix

$$A := \left[\underbrace{a^1}_{a^1}, \underbrace{Sa^1}_{a^2}, Sa^2, \dots, \underbrace{Sa^{d-1}}_{a^d} \right] = \left[a^1, Sa^1, S^2 a^1, \dots, S^{d-1} a^1 \right]$$

is invertible.

(3) Show that $S := ACA^{-1}$ is a matrix representation of the tensor S.

or equivalently show that $SA = AC$.

(4) It is enough to find the relation between characteristic polynomial of S and C.

problem 4.

- A symmetric matrix S can be written of the form $S = F^T \Omega_S F$ where Ω_S is a diagonal matrix with eigenvalues of S on the diagonal. and F is the orthogonal matrix $(FF^T = I)$ which has the corresponding eigenvectors as columns.

- Observe that $S^m = F \Omega_S^m F^T$.
- Computing the powers of a diagonal matrix is easy. :)

- For an analytic function f we have:

$$f(S) = F f(\Omega_S) F^T.$$

- $f(\Omega_S)$ is just f acting on the diagonals of Ω_S.

- $S^{1/2} := F \sqrt{\Omega_S}^{\,c} F^T$

- Finding square roots of a diagonal matrix is easy :>

e.g. $A = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$ $\quad A^{1/2} = \begin{bmatrix} \pm 2 & 0 \\ 0 & \pm i\sqrt{2} \end{bmatrix}$

A has 4 square roots.

Problem 5.

Remember the definitions
$$\begin{cases} I_1 = -\operatorname{tr}(S) \\ I_2 = \dfrac{(\operatorname{tr} S)^2 - \operatorname{tr}(S^2)}{2} \\ I_3 = -\det S \end{cases}$$

and $QQ^T = Q^T Q = I$.

- $\det(AB) = \det(BA)$
- $tr(AB) = tr(BA)$

Problem 6

If rank $(A \vdots b)$ is 4 then the system has no solution. So find the unique values $(s,t) = (a,b)$ for which $(A \vdots b)$ is not full rank. Then you know that the system has solutions only if $s = a$ or $t = b$.

Consider 3 cases: $s = a, t = b$
$s = a, t \neq b$ and check them.
$s \neq a, t = b$

Problem 7.

Change of Basis Matrix:

Let $v_1, v_2, \ldots, v_n$ be a basis for the vector space V. Let $v_1', v_2', \ldots, v_n'$ be another basis. You can write

$$v_1' = \alpha_{11} v_1 + \alpha_{21} v_2 + \alpha_{31} v_3 + \cdots + \alpha_{n1} v_n$$
$$\vdots$$
$$v_n' = \alpha_{1n} v_1 + \alpha_{2n} v_2 + \alpha_{3n} v_3 + \cdots + \alpha_{nn} v_n$$

Now form the matrix

$$Q_{vv'} = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \alpha_{21} & \ddots & \vdots \\ \vdots & & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{bmatrix}$$

This is called the matrix of change of basis from v to v'.

This means $\begin{pmatrix} a_1' \\ \vdots \\ a_n' \end{pmatrix}_{v'} = Q_{v'v} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}_{v}$

NOTE: $Q_{v'v} = Q_{vv'}^{-1}$.

$*$ let $T: V \longrightarrow W$ be a tensor and

$\left.\begin{matrix} e_1, \dots, e_n \\ e_1', \dots, e_n' \end{matrix}\right\} \rightarrow$ two basis for V and

$\left.\begin{matrix} \varepsilon_1, \dots, \varepsilon_m \\ \varepsilon_1', \dots, \varepsilon_m' \end{matrix}\right\} \rightarrow$ two basis for W and

Let $M_T^{e\varepsilon}$ be the matrix of T for basis e for V and ε for W.

The matrix $M_T^{e'\xi'} = Q_{\xi'\xi} \, M_T^{e\xi} \, Q_{ee'}$.

In Problem 7

let $f_1' = f^1$

$$f_2' = f^1 + f^2$$

$$f_3' = f^1 + f^2 - f^3$$

$$Q_{ff'} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}_{3\times 3} \quad - - -$$