Solutions of problem sheet 1 / OCT 2019 / Leipzig Uni

① ⓐ WTS $\|A\|_{op} = \overbrace{\sup_{\|v\|=1} \|Av\|}^{\beta}$.

by definition $\|A\|_{op} = \underbrace{\sup_{v \in V\setminus\{0\}} \frac{\|Av\|}{\|v\|}}_{\alpha} \in [0,\infty)$ for $A \in \mathcal{L}(V)$

So we need to show that $\alpha = \beta$.

$\boxed{\alpha \geq \beta}$ bcz $\sup_{v\in V\setminus\{0\}} \frac{\|Av\|}{\|v\|} \geq \sup_{\|v\|=1} \frac{\|Av\|}{\|v\|}$ $\left(\{v \in V \text{ s.t. } \|v\|=1\} \subseteq \{v\in V \text{ s.t. } v \neq 0\}\right)$ ~~(and $\|A\|_{op} < \infty$) (?)~~

$\boxed{\alpha \leq \beta}$ bcz $\|Av\| \leq \beta \quad \forall v \in V$ s.t $\|v\| = 1$

$\Longrightarrow \|A\left(\frac{v}{\|v\|}\right)\| \leq \beta \quad \forall v \neq 0$ since $\|\frac{v}{\|v\|}\| = 1$

$\Longrightarrow \frac{\|Av\|}{\|v\|} \leq \beta \quad \forall v \neq 0 \Longrightarrow \alpha \leq \beta$. ✓

WTS $\|A.B\|_{op} \leq \|A\|_{op} \cdot \|B\|_{op} \quad \forall A, B \in \mathcal{L}(V)$.

We use the ineq. $\|A.v\| \leq \|A\|_{op} \|v\| \quad \forall A \in \mathcal{L}(V)$ $v \in V$

this is true by the def. $\|A\|_{op} = \sup_{v\neq 0} \frac{\|Av\|}{\|v\|}$ so

$\forall v \in V$, $v \neq 0$: $\frac{\|Av\|}{\|v\|} \leq \|A\|_{op} \Rightarrow \|Av\| \leq \|A\|_{op} \|v\|$ (this obviously holds for $v=0$ as well.)

$\|AB\|_{op} = \sup_{\|v\|=1} \|ABv\| \leq \sup_{\|v\|\leq 1} \|A\|_{op} \cdot \|Bv\|$

$= \|A\|_{op} \sup_{\|v\|=1} \|Bv\| = \|A\|_{op} \|B\|_{op}$. ✓

(b) Let $B := I + A + A^2 + A^3 + \cdots = I + \sum_{i=1}^{\infty} A^i$

WTS $\sum_{i=0}^{\infty} A^i$ is converging and $\sum_{i=0}^{\infty} A^i$ is the inverse of $I - A$.

From Background: $\sum_{n=0}^{\infty} r^n$ is convergence for $|r|<1$ to $\frac{1}{1-r}$

so $\sum_{n=0}^{\infty} \|A\|^n$ is convergence for $\|A\| < 1$.

so $I + A + A^2 + \cdots$ is convergence for $\|A\| < 1$

and $(I-A)(I + A + A^2 + \cdots) = (I + \cancel{A} + \cancel{A^2} + \cancel{\cdots}) - (\cancel{A} + \cancel{A^2} + \cancel{\cdots}) = I$.

and $(I + A + A^2 + \cdots)(I - A) = I$ ✓ (unnecessary)

WTS $\left.\begin{matrix} A \text{ inv.} \\ \|B\| < \|A^{-1}\|^{-1} \end{matrix}\right\} \Rightarrow A + B$ inv.

if $\|A\| < 1$

:)

we have shown that $I - C$ is invertible for $\|C\| < 1$,

let $C = B \cdot A^{-1}$, ..

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Now since $\|B\| < \|A^{-1}\|^{-1} \Rightarrow \|B\|\|A^{-1}\| < 1$

from (a) $\|C\| = \|B \cdot A^{-1}\| \leq \|B\|\|A^{-1}\| < 1$

so $I + B \cdot A^{-1}$ is invertible with inverse $\sum_{n=0}^{\infty} (-BA^{-1})^n$

$\Rightarrow$

$(I + B \cdot A^{-1}) \cdot A = A + B$ is invertible.

(c) let $V = \mathbb{R}^2$ $\|(a,b)\| = \max\{|a|,|b|\}$ let $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ $A^{-1} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

$\|A\|_{op} = \sup_{\|v\|=1} \|Av\| = \sup_{\|v\|=1} \{\max\{|v_1|, |2v_2|\}\} = \boxed{2}$ i.e. $\max\{|v_1|, |v_2|\} = 1$ $\|A^{-1}\|_{op} = \boxed{1} \Rightarrow \|A\|^{-1} \neq \|A^{-1}\|$

2 (a) WTS $GL(n, \mathbb{R}) \subset M(n \times n, \mathbb{R})$ is an open and dense subset.

Remember $M(n \times n, \mathbb{R}) \cong \mathbb{R}^{n^2}$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \to (a_{11}, \dots, a_{nn})$$

let $f : M(n \times n, \mathbb{R}) \longrightarrow \mathbb{R}$

$A \longmapsto \det A$

this map is continuous, so the inverse image of an open subset of $\mathbb{R}$ is open in A.

$\Rightarrow GL(n, \mathbb{R}) = \mathbb{R} \setminus \{0\}$ is open in $M(n \times n, \mathbb{R})$.

To show density: an easy and fast solution:

we want to show for any matrix $A \in M(n \times n, \mathbb{R})$, and $\forall \varepsilon >0$
$\exists B \in M(n \times n, \mathbb{R})$ s.t. B is in $GL(n, \mathbb{R})$ and $\|A - B\| < \varepsilon$

~~absolute value~~ ~~nonnegative~~ positive

→ let $0 < \varepsilon <$ All the positive eigenvalues of A $\Rightarrow A - \varepsilon I$ is invertible (otherwise $\det(A - \varepsilon I) = 0$, which is a contradiction with the choice of ε.
ε is an eigenvalue.

→ let $p : \mathbb{R} \longrightarrow \mathbb{R}$ so $p(t)$ is a univariate function
$t \longmapsto \det(A - tI)$ of deg at most n so

so $p(t)$ has at most n distinct roots. so one can choose t arbitrarily small s.t. $\det(A - tI) \neq 0$ $\|A - A + tI\| = \|tI\|_{op} = \sup_{\|v\| \leq 1} \|tI \cdot v\| = |t| \sup_{\|v\|=1} \|v\| = |t|$

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(b) $f : M(n\times n, \mathbb{R}) \to \mathbb{R}$ cont. WTS

$f(CAC^{-1}) = f(A)$ $\forall A \in M(n\times n,\mathbb{R})$ $\forall C \in GL(n,\mathbb{R})$ $\iff$ $f(AB) = f(BA)$ $\forall A,B \in M(n\times n, \mathbb{R})$

$(\Leftarrow)$ trivial direction: if C inv. and $f(AC) = f(CA) \overset{\leftarrow C^{-1}}{\Longrightarrow} f(\underbrace{ACC^{-1}}_{f(A)}) = f(CAC^{-1})$ ✓

$(\Rightarrow)$ let $\varepsilon > 0$ s.t. $A - \varepsilon I$ is invertible. (by (a) such ε exists).

$f((A-\varepsilon I)B(A-\varepsilon I)^{-1}) = f(B) \Longrightarrow f((A-\varepsilon I)B) = f(B(A-\varepsilon I))$

$\Rightarrow f(AB - \varepsilon B) = f(BA - \varepsilon B)$

from (a) we know that we can choose $\varepsilon > 0$ arbitrarily small,

Since f is continuous

$\lim_{\varepsilon \to 0} f(AB - \varepsilon B) = f(AB)$ $\lim_{\varepsilon \to 0} f(BA - \varepsilon B) = f(BA)$

$\to f(AB) = f(BA)$ $\forall A, B \in M(n\times n, \mathbb{R})$

Now we show $tr(AC) = tr(CA)$ $\forall C \in GL(n,\mathbb{R})$ $A \in M(n\times n, \mathbb{R})$

let $N = AC$ i.e. $n_{ij} = \sum_{k=1}^{n} a_{ik} c_{kj}$

and $L = CA$ i.e. $l_{ij} = \sum_{k=1}^{n} c_{ik} a_{kj}$

WTS $tr(N) = tr(L)$.

$tr(N) = \sum_{i=1}^{n} n_{ii} = \sum_{i=1}^{n} \left(\sum_{k=1}^{n} a_{ik} c_{ki} \right) = \left(\sum_{i=1}^{n} \left(\sum_{k=1}^{n} c_{ki} a_{ik} \right) \right) =$

$\sum_{k=1}^{n} \left(\sum_{i=1}^{n} c_{ki} a_{ik} \right) = \sum_{k=1}^{n} l_{kk} = tr(L)$ ✓.

So $tr(CAC^{-1}) = tr(A)$ $\forall A \in M(n\times n, \mathbb{R})$ $C \in GL(n, \mathbb{R})$.

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(c) WTS $O(n,\mathbb{R})$ is compact.

let $\varphi: M(n\times n, \mathbb{R}) \longrightarrow M(n\times n, \mathbb{R})$

$$A \longmapsto AA^T$$

$O(n,\mathbb{R}) = \varphi^{-1}(I)$ and thus is closed. (I is a clsd pt in).

we need to show that $O(n,\mathbb{R})$ is bdd. For that let

$$B = \{ A \in M(n\times n, \mathbb{R}) \text{ st. } a_{ij} \leq 1 \ \forall i,j \}$$

This is bdd subset of $M(n\times n, \mathbb{R})$ (think of the isom. ~~wh~~ with $\mathbb{R}^{n^2}$

so we have $(a_{11}, a_{12}, \dots, a_{nn})$ s.t. $a_{ij} \leq 1$) ✓

this is the ~~max~~ norm $\|v\| = \max |v_i|$

WTS $O(n,\mathbb{R}) \subseteq B$

let $A \in O(n,\mathbb{R})$

$$AA^T = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{n1} & & & a_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{n1} \\ a_{12} & & \\ \vdots & & \\ a_{1n} & & a_{nn} \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

$$\sum_{k=1}^{n} a_{ik} = 1 \quad \forall i = 1, \dots, n \qquad \text{so} \quad a_{ij} \leq 1 \quad \forall i,j \in \{1, \dots, n\}$$

$\Rightarrow O(n,\mathbb{R}) \subseteq B$ and $O(n,\mathbb{R})$ is bdd.

clsd + bdd $\Rightarrow$ compact. $\quad \mathbb{R}^{n^2}$ is a metric sp.

$M(n\times n, \mathbb{R})$

3 (a) $f: X \longrightarrow Y$ homeom. of top sp. — i.e. f is bij.

f, f^{-1} both continuous.

X path connected $\Rightarrow Y$ is path connected.

~~$\$$~~ to show Y is path connected $\forall y_1, y_2 \in Y$ we need to find a cont. map ~~$\$$~~

$\gamma: [0,1] \to Y$ s.t. $\gamma(0) = y_1$ and $\gamma(1) = y_2$.

f is a bij so $\exists!\, x_1 \in X$ s.t. $f(x_1) = y_1$ & $f(x_2) = y_2$ and since X is
$x_2 \in X$

Path. connected $\exists \tilde{\gamma}: [0,1] \to X$ s.t. $\tilde{\gamma}(0) = x_1$ and $\tilde{\gamma}(1) = x_2$, take $\gamma := f \circ \tilde{\gamma}$

$\tilde{\gamma}: [0,1] \to Y$ cont. ✓

$\tilde{\gamma}(0) = f(\gamma(0)) = y_1$ ✓

$\tilde{\gamma}(1) = f(\gamma(1)) = y_2$

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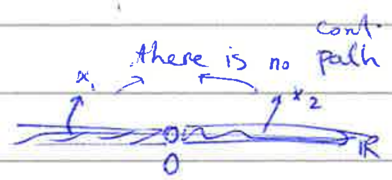
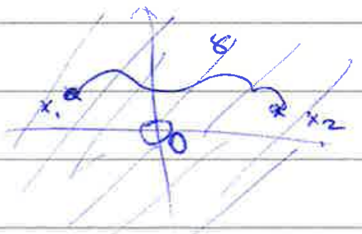
(b) WTS $\mathbb{R}^2$ and $\mathbb{R}$ are not homeom.

we can use (a) and show that $\mathbb{R}^2 \setminus \{0\}$ is path connected and $\mathbb{R} \setminus \{0\}$ is not.



$\mathbb{R}^2$



step 1. $f: X \to Y$ homeom. of top sp. $U \subseteq X$ any subset
$\Rightarrow f|_U : X \to f(U)$ " " " " with subsp. top.

step 2. let $U = \mathbb{R} \setminus \{x\}$ for some $x \in \mathbb{R}$. So if $\exists \varphi : \mathbb{R} \to \mathbb{R}^2$ homeom. then $\varphi|_U : \mathbb{R} \setminus \{x\} \to \mathbb{R}^2 \setminus \varphi(x)$ is also a homeom.

step 3. prove that ~~continuous image of a connected space is connected.~~ X path-connected $\Rightarrow$ X connected.

~~step 4. if $\gamma : [0,1] \to \mathbb{R} \setminus \{x\}$ is continuous then the image of $[0,1]$ has to be connected.~~

step 4. $\mathbb{R} \setminus \{x\}$ is not path connected. bcz is not connected.

step 5. $\mathbb{R}^2 \setminus \{\varphi(x)\}$ is path connected.

~~step 7.~~ conclusion: such φ can not exist
$\Rightarrow \mathbb{R}$ is not homeom. to $\mathbb{R}^2$.

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bij / f, f^{-1} both continuous.

step 1/. $X \to Y$ homeom. , $U \subseteq X$ any subset

- $f|_U : U \to f(U)$ is clearly a bijection. ✓

- restriction of a cont. map is again cont.

- $(f|_U)^{-1} = f^{-1}|_{f(U)}$ So again is the restriction of a cont. map and so is cont.

check the proof on your own.

$\Rightarrow f|_U : U \to f(U)$ is a homeomorphism, where U and $f(U)$ both have subspace topology.

step 3/ X path connected $\Rightarrow$ X connected. have seen the sin example $\cancel{\Leftarrow}$

Let X be disconnected, i.e. $\exists U_1, U_2$ open in X s.t. $\begin{cases} U_1 \neq \emptyset, U_2 \neq \emptyset \\ U_1 \cap U_2 = \emptyset \\ U_1 \cup U_2 = X \end{cases}$

let $x_1 \in U_1$ and $x_2 \in U_2$

we will use the fact that $[0,1]$ is connected and preimage of a disconnected space is disconnected to get to a contradiction.

since U_1, U_2 are non-empty.

If X is path-connected $\exists \gamma : [0,1] \to X$ continuous so that

$$\gamma(0) = x_1$$
$$\gamma(1) = x_2$$

$$[0,1] = \gamma^{-1}(X) = \gamma^{-1}(U_1 \cup U_2) = \underbrace{\gamma^{-1}(U_1)}_{\ni 0} \cup \underbrace{\gamma^{-1}(U_2)}_{\ni 1}$$

so they are non-empty.

also $\gamma^{-1}(U_1) \cap \gamma^{-1}(U_2) = \gamma^{-1}(U_1 \cap U_2) = \gamma^{-1}(\emptyset) = \emptyset$

$\Rightarrow [0,1]$ is disconnected ↯. $\Rightarrow$ X is connected.

step 4. $\mathbb{R}\setminus\{x\}$ is disconnected bcz $\mathbb{R}\setminus\{0\} = \overbrace{\left[(\mathbb{R}\setminus\{x\}) \cap \underbrace{(-\infty, x)}_{\text{open in }\mathbb{R}}\right]}^{\text{open in } \mathbb{R}\setminus\{0\}} \cup \underbrace{\left[(\mathbb{R}\setminus\{x\}) \cap (x, +\infty)\right]}_{\text{open in } \mathbb{R}\setminus\{0\}}$

they are both non-empty and their intersection is empty.

so $\mathbb{R}\setminus\{x\}$ is not path-connected.

step 5. $\mathbb{R}^2\setminus\{\varphi(x)\}$ is path-connected:

we show a more general statement, $\mathbb{R}^2\setminus A$ where $A \subseteq \mathbb{R}^2$ is a countable set is path connected.

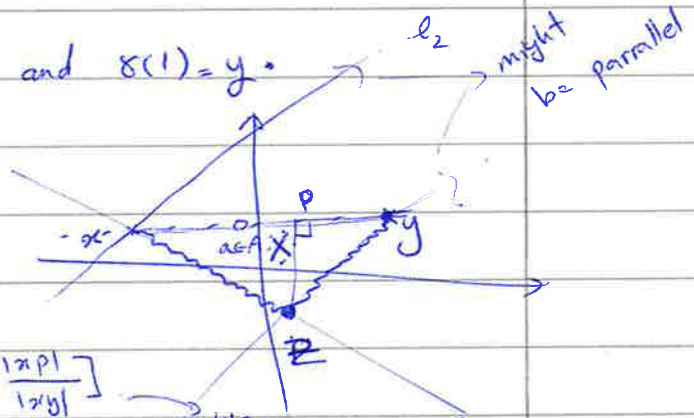
pf. let $x, y \in \mathbb{R}^2\setminus A$, since A is countable, there are infinitly many line through x that do not intersect A. Similarly there are ~ ~ ~ ~ y ~ ~ ~ ~ ~. Pick 2 of them that are through x (l_1 and l_2) and one that is through y (l_3) ~~the lines~~. l_3 intersects at least one of the two lines l_1 and l_2. (let say $l_3 \cap l_1 = \{z\}$)

then take the union of l_3 and l_2, they make a ~~path~~ path

$\gamma : [0,1] \to \mathbb{R}^2\setminus\{A\}$ s.t. $\gamma(0) = x$ and $\gamma(1) = y$.

let $p \in \mathbb{R}^2$ be as in the picture:

γ can be defined as follows:

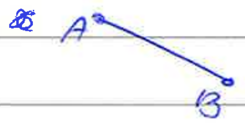


$$\gamma(t) = \begin{cases} \frac{|xy|}{|xp|} \cdot \left(\frac{|xp|}{|xy|} - t \right) x + t z & t \in \left[0, \frac{|xp|}{|xy|}\right] \\ \frac{|xy|}{|yp|} \cdot \left((1-t) z + \left(t - \frac{|xp|}{|xy|} y\right)\right) & t \in \left(\frac{|xp|}{|xy|}, 1\right] \end{cases}$$

we map this interval to xz

and this to zy.

the main idea is that to map $[\cdot,1]$ to the ~~interval~~ line segment



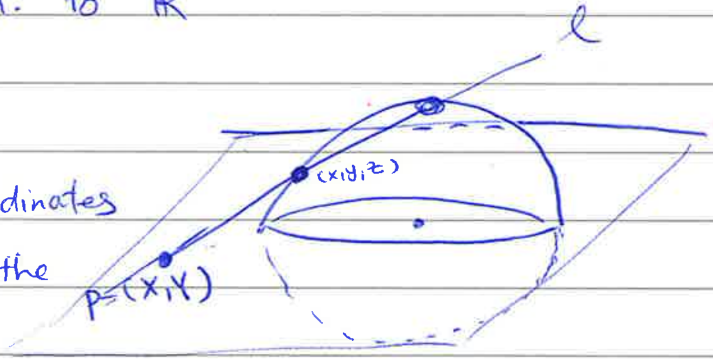
we do $t \longmapsto (1-t)A + tB$

now we do some rescaling and shifts to find our desired map. /

$\blacksquare$

(c) $S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1\}$

WTS $S^2 \setminus \{(0,0,1)\}$ is homeom. to $\mathbb{R}^2$



~~First we~~

let (x,y,z) be the coordinates of $\mathbb{R}^3$ and (X,Y) be the coordinates of $\mathbb{R}^2$.

let $\varphi: S^2 \setminus \{(0,0,1)\} \longrightarrow \mathbb{R}^2$ be the one that maps a point (x,y,z) to the point $P=(X,Y)$ in $\mathbb{R}^2$ s.t. P is the intersection of the line passing through $(0,0,1)$ and (x,y,z) and $\mathbb{R}^2$. you can simply see that this a $\boxed{\text{bijection}}$. Lets find the coordinates of (X,Y), i.e. lets find the defining equations of the map φ.

$$l: (1-t)(0,0,1) + t(x,y,z) = (tx, ty, 1-t+tz) \quad (*)$$

intersection of l with the plane $\mathbb{R}^2_{(X,Y,0)}$ means we need to set the last cordinate to be zero.

$$\Rightarrow (1-t) + tz = 0 \Rightarrow t = \frac{1}{1-z}$$

$$\Rightarrow (X,Y) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right) \quad \text{from } (*).$$

So $\varphi: S^2 \setminus \{(0,0,1)\} \longrightarrow \mathbb{R}^2$

$$(x,y,z) \longmapsto \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$

φ is cont. on the sphere without the North pole. (Note that the north pole is the only pt with $z=1$).

~~Also~~ Also we can find the inverse map:

$$\psi : \mathbb{R}^2 \longrightarrow S^2 \setminus \{(0,0,1)\}$$
$$(X,Y) \longmapsto (x,y,z)$$

we need to find the intersection point of the line through $(X,Y,0)$ and $(0,0,1)$, with the sphere. as before:

$$\ell' : (1-t)(0,0,1) + t(X,Y,0) = (tX, tY, (1-t))$$

to be on sphere:

$$(tX)^2 + (tY)^2 + (1-t)^2 = t^2(X^2+Y^2) + \cancel{1} + t^2 - 2t = \cancel{1}\ 0$$

$t=0$ gives $(0,0,1)$, so we can cancel it from the equation since we are looking for the other pt.

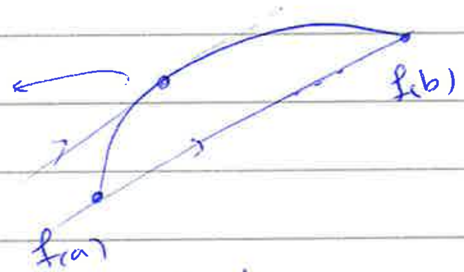
$$t(X^2+Y^2)+t-2=0 \Rightarrow t = \frac{2}{X^2+Y^2+1}$$

$$\Rightarrow (x,y,z) = \left(\frac{2X}{X^2+Y^2+1}, \frac{2Y}{X^2+Y^2+1}, \frac{X^2+Y^2-1}{X^2+Y^2+1} \right)$$

ψ is cont. on $\mathbb{R}^2$ and one can check that $\psi \circ \varphi = 1_{S^2\setminus\{(0,0,1)\}}$ and $\varphi \circ \psi = 1_{\mathbb{R}^2}$ so $\psi = \varphi^{-1}$ and thus φ is a homeom. ✓

(d) Mean Value theorem: Let $f:[a,b]\to\mathbb{R}$ be a cont. func. and differentiable on (a,b), $\exists c \in (a,b)$ s.t.

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$



to show that $f:[0,1]\to[0,1]$ ~~Banach unique~~ is a contraction we need to find $0 \le k < 1$ s.t. $\forall x,y \quad 0\le x,y \le 1 \quad |f(x)-f(y)| \le k|x-y|$

by mean value thm $\forall x,y \in [0,1]$, $x<y$, $\exists c \in (x,y)$ s.t. $|f(x)-f(y)| = |f'(c)(x-y)|$

let $k = \sup_{c\in[0,1]} |f'(c)|$, we have $|f(x)-f(y)| \le \overset{k}{\cancel{|f'(c)|}}|x-y|$

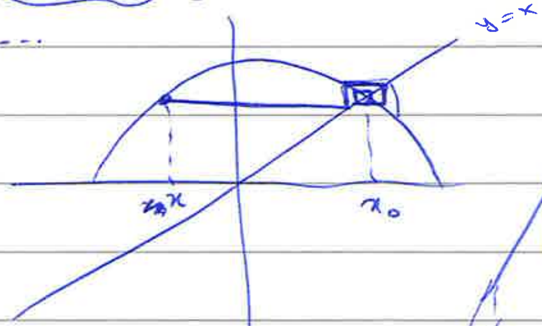
by assumption $|f(c)|<1 \quad \forall c \in [-1,1] \Rightarrow k<1$

Banach fixed thm implies that f has a unique fixed pt.

Attractive fixed point is a fixed pt x_0 of f s.t. $\forall x$ close enough to x_0 the iterated function sequence

$x, f(x), f(f(x)), \ldots$

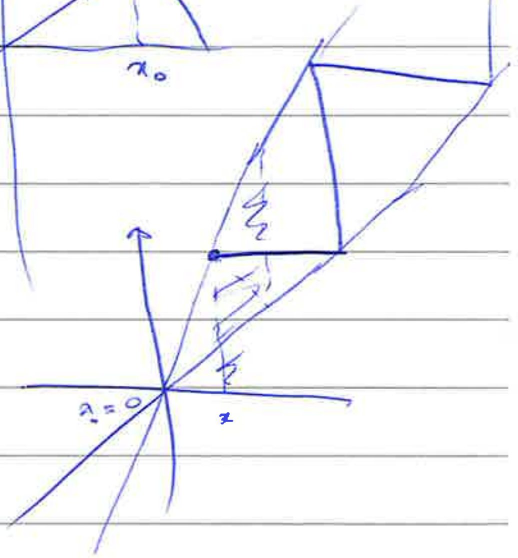
converges to x_0.



not all fixed pts are attractive.

$f(x) = 2x$

0 is fixed pt. but not attraction pt.



~~Banach fixed thm's assumption~~

if f cont., continuously differentiable and $|f'(x)|<1$ attraction is garanteed :)

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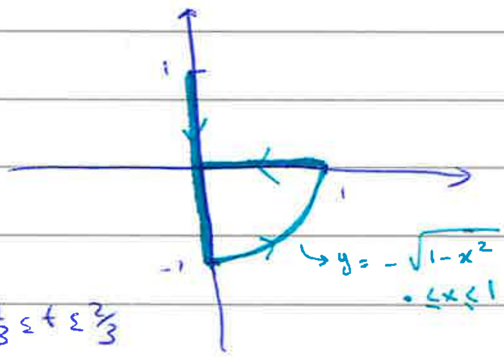
4 lets plot $C \subseteq \mathbb{R}^2$



Note that

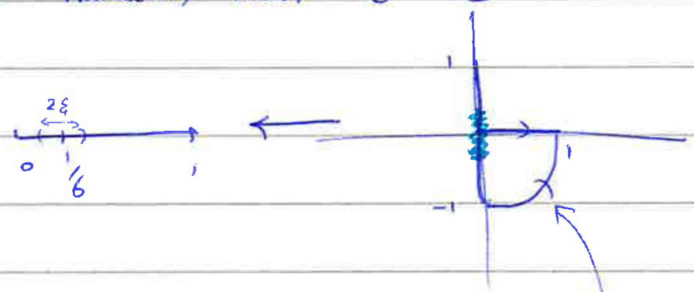
$(\sin \frac{\pi}{2}(3t-1), -\cos\frac{\pi}{2}(3t-1)) \quad \frac{1}{3} \leq t \leq \frac{2}{3}$

are pts $y = -\sqrt{1-x^2} \qquad 0 \leq x \leq 1$

each piece of the function is cont. and it is easy to check that it is also continuous on the boundaries. So γ is cont.

γ is also a bijection bcz each piece is a bij and $\forall t_1 \neq t_2 \in (-1,1)$ $\gamma(t_1) \neq \gamma(t_2)$

~~理/A~~ If γ is a homeom, then $\gamma^{-1} : C \longrightarrow (0,1)$ is a cont. function



Thus preimage of $\underbrace{(\frac{1}{6}-\varepsilon, \frac{1}{6}+\varepsilon)}_{U_\varepsilon}$ for $0<\varepsilon<\frac{1}{6}$ has to be open in C ~~but the~~ (this set is shown by light blue). but this can not be true bcz U_ε can not be the intersection of an open ball and C; any such intersection will also ~~contain~~ contain points from $(3-3t, 0)$ $\frac{2}{3} \leq t < 1$.

So $(0,1)$ is not homeomorphic to C.

5. $$S(\gamma+tv) - S(\gamma) = \frac{1}{2}\int_a^b |(\dot{\gamma}+t\dot{v})(s)|^2 ds - \frac{1}{2}\int_a^b |\dot{\gamma}(s)|^2 ds$$

let $|\cdot|^2 = \underbrace{\langle\ ,\ \rangle}_{\text{inner product, (dot product)}}$

$$= \frac{1}{2}\int_a^b \left(\langle \dot{\gamma}+t\dot{v}_{(s)}, \dot{\gamma}+t\dot{v}_{(s)}\rangle - \langle \dot{\gamma}(s), \dot{\gamma}(s)\rangle\right) ds$$

$$= \frac{1}{2}\int_a^b \left(\cancel{\langle \dot{\gamma}(s), \dot{\gamma}(s)\rangle} + 2t\langle \dot{\gamma}_{(s)}, \dot{v}_{(s)}\rangle + t^2\langle \dot{v}_{(s)}, \dot{v}_{(s)}\rangle - \cancel{\langle \dot{\gamma}(s), \dot{\gamma}(s)\rangle}\right) ds$$

$$= t\int_a^b \langle \dot{\gamma}(s), \dot{v}(s)\rangle ds + t^2 \int \langle \dot{v}(s), \dot{v}(s)\rangle ds$$

$$\Rightarrow \lim_{t\to 0} \frac{S(\gamma+tv) - S(\gamma)}{t} = \int_a^b \langle \dot{\gamma}(s), \dot{v}(s)\rangle ds = \int_a^b \left(\sum_{i=1}^n (\dot{\gamma}_i \dot{v}_i)(s)\right) ds$$

$$= \sum_{i=1}^n \int_a^b (\dot{\gamma}_i \dot{v}_i)(s) ds = \sum_{i=1}^n \left(\underbrace{[(\dot{\gamma}_i v_i)(s)]_a^b}_{0} - \int_a^b (\ddot{\gamma}_i v_i)(s) ds\right) = -\int_a^b \langle \ddot{\gamma}(s), v(s)\rangle ds$$

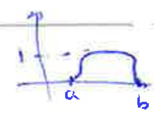
use integration by parts

0 bcz $v_i(a) = v_i(b) = 0$

Since we know that all such directional derivatives are zero $\Rightarrow \int_a^b \langle \ddot{\gamma}(s), v(s)\rangle ds = 0$

let $v(s) = B_\varepsilon(s)\ddot{\gamma}(s)$ where $B_\varepsilon(s) : [a,b] \to \mathbb{R}$ $B_\varepsilon(s) = \begin{cases} 1 & [a+\varepsilon, b-\varepsilon] \\ 0 & \text{otherwise} \end{cases}$

sufficiently small $\leftarrow \varepsilon s.$



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$\hookrightarrow$ * check the existence!

$$0 = \int_a^b \langle \ddot{\gamma}(s), v(s)\rangle ds = \int_a^b \langle \ddot{\gamma}(s), \ddot{\gamma}(s) B_\varepsilon(s)\rangle ds = \int_a^b |\ddot{\gamma}(s)|^2 B_\varepsilon(s) ds = 0$$

Since ε can be arbitrary small $\Rightarrow \ddot{\gamma}(s) = 0$