Problem Sheet 10

Due Date: 04.05.2020, 12:00 UTC+2 (CEST)

Problem 1. [**1** pt] Show that for any two tensors $S : V \to V$, $T : V \to V$, with V being a finite-dimensional linear space

$$tr(ST) = tr(TS)$$

Problem 2. [**1+1+1** pts] Let e, f be orthogonal unit vectors in $\mathbb{R}^3$. (i) Show that

$$(\mathbb{1} - e \otimes e)v = -e \times (e \times v)$$

for any $v \in \mathbb{R}^3$. (ii) What does $(\mathbb{1} - e \otimes e)$ mean geometrically? (iii) What does $e \otimes e + f \otimes f$ mean geometrically?

Problem 3. [**2** pts] Let us denote by *sym* the symmetric part of a tensor. Show

$$sym(A^T BA) = A^T sym(B)A,$$

where A, B are tensors $\mathbb{R}^d \to \mathbb{R}^d$.

Problem 4. [**3+2+3** pts] Consider an arbitrary skew-symmetric tensor $\Omega : \mathbb{R}^3 \to \mathbb{R}^3$.
(i) Show that Ω be represented by a vector ω as follows:

$$\forall_{v \in \mathbb{R}^3} \quad \Omega v = \omega \times v.$$

(ii) Next, show that cofactor of Ω equals $\omega \otimes \omega$.
(iii) Prove that $\mathbb{1} + \Omega$ is always invertible.

Problem 5. [**1** pts] Show that for any symmetric tensor $S : \mathbb{R}^d \to \mathbb{R}^d$ and antisymmetric tensor $W : \mathbb{R}^d \to \mathbb{R}^d$ it holds

$$S : W = 0.$$

Problem 6. [**4** pts] Let $f_{i,j}$ be differentiable functions on $\mathbb{R}$, $i, j = 1, \ldots, n$. Define F to be the $n \times n$-matrix with entries $f_{i,j}$. Show that

$$(detF)' = \sum_{i=1}^{n} det \begin{bmatrix} f_{1,1} & \cdots & f_{1,n} \\ \vdots & \cdots & \vdots \\ f_{i-1,1} & \cdots & f_{i-1,n} \\ f'_{i,1} & \cdots & f'_{i,n} \\ f_{i+1,1} & \cdots & f_{i+1,n} \\ \vdots & \cdots & \vdots \\ f_{n,1} & \cdots & f_{n,n} \end{bmatrix}$$