Series 5

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1) $(x,y,z) = (r\sin\vartheta\cos\varphi,\ r\sin\vartheta\sin\varphi,\ r\cos\vartheta)$

$\vartheta \in [0,\pi]$!

$\varphi \in [0,2\pi]$

a) $\alpha = x\,dx + y\,dy + z\,dz$

$= d\left(\frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2\right)$

$x^2+y^2+z^2 = r^2\sin^2\vartheta\cos^2\varphi + r^2\sin^2\vartheta\sin^2\varphi + r^2\cos^2\vartheta$

$= r^2$

$\Rightarrow \quad \alpha = d\left(\frac{1}{2}r^2\right) = \boxed{r\,dr}$ //

$$(dx, dy, dz) = (dr, d\vartheta, d\varphi)\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \vartheta} & \frac{\partial y}{\partial \vartheta} & \frac{\partial z}{\partial \vartheta} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} & \frac{\partial z}{\partial \varphi} \end{pmatrix}$$

$$= (dr, d\vartheta, d\varphi)\underbrace{\begin{pmatrix} \sin\vartheta\cos\varphi & \sin\vartheta\sin\varphi & \cos\vartheta \\ r\cos\vartheta\cos\varphi & r\cos\vartheta\sin\varphi & -r\sin\vartheta \\ -r\sin\vartheta\sin\varphi & r\sin\vartheta\cos\varphi & 0 \end{pmatrix}}_{=:A}$$

$\Rightarrow \gamma = dx\wedge dy\wedge dz = \det A \cdot dr\wedge d\vartheta\wedge d\varphi$

$= \boxed{r^2\sin\vartheta\, dr\wedge d\vartheta\wedge d\varphi}$ //

A is orthogonal $\Rightarrow$ $A^{-1} = \begin{pmatrix} \sin\vartheta\cos\varphi & r\cos\vartheta\cos\varphi & -r\sin\vartheta\sin\varphi \\ \sin\vartheta\sin\varphi & r\cos\vartheta\sin\varphi & r\sin\vartheta\cos\varphi \\ \cos\vartheta & -r\sin\vartheta & 0 \end{pmatrix}$

$$= \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial \vartheta}{\partial x} & \frac{\partial \varphi}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial \vartheta}{\partial y} & \frac{\partial \varphi}{\partial y} \\ \frac{\partial r}{\partial z} & \frac{\partial \vartheta}{\partial z} & \frac{\partial \varphi}{\partial z} \end{pmatrix}$$

$$\partial_x = \frac{\partial r}{\partial x}\partial_r + \frac{\partial \vartheta}{\partial x}\partial_\vartheta + \frac{\partial \varphi}{\partial x}\partial_\varphi$$

$$\Rightarrow \begin{pmatrix}\partial_x\\ \partial_y\\ \partial_z\end{pmatrix} = A^{-1}\begin{pmatrix}\partial_r\\ \partial_\vartheta\\ \partial_\varphi\end{pmatrix}$$

$$\Rightarrow (x,y,z)\begin{pmatrix}\partial_x\\ \partial_y\\ \partial_z\end{pmatrix} = (r\sin\vartheta\cos\varphi, r\sin\vartheta\sin\varphi, r\cos\vartheta)\, A^{-1}\begin{pmatrix}\partial_r\\ \partial_\vartheta\\ \partial_\varphi\end{pmatrix}$$

$$= (r, 0, 0)\begin{pmatrix}\partial_r\\ \partial_\vartheta\\ \partial_\varphi\end{pmatrix}$$

$$\Rightarrow x\partial_x + y\partial_y + z\partial_z = r\partial_r$$

Have $\beta = x\, dy\wedge dz - y\, dx\wedge dz + z\, dx\wedge dy$

$$= (x\partial_x + y\partial_y + z\partial_z) \lrcorner (dx\wedge dy\wedge dz)$$

$$= r\partial_r \lrcorner \gamma = r\partial_r \lrcorner r^2\sin\vartheta\, dr\wedge d\vartheta\wedge d\varphi$$

$$= \boxed{r^3\sin\vartheta\, d\vartheta\wedge d\varphi}$$

Observe $d\beta = 3\gamma$ in both coordinate systems as a test for the correctness

Note $X \lrcorner \omega \overset{\text{def.}}{=} i_X\omega$

1 (b)

Ansatz $X = f(r)\,\partial_r$

$$\omega = \iota_X \mathrm{vol} = r^2 f(r) \sin\vartheta \, d\vartheta \wedge d\varphi$$

$$\Rightarrow d\omega = \left(2r f(r) + r^2 f'(r)\right) \sin\vartheta \, dr \wedge d\vartheta \wedge d\varphi$$

$$= 0 \iff f \text{ solves}$$

$$f' = -\frac{2}{r} f$$

$$\iff (\log f)' = \left(\log \frac{1}{r^2}\right)'$$

$$\iff f(r) = \frac{\text{const}}{r^2}$$

Let $\omega = \sin\vartheta \, d\vartheta \wedge d\varphi$

$$\int_{S^2_{r_0}} \omega = \int_0^\pi \sin\vartheta \, d\vartheta \int_0^{2\pi} d\varphi = (-\cos\vartheta)\Big|_0^\pi \cdot 2\pi$$

$$= 4\pi$$

If $\omega = d\eta$ for some $\eta \in \Omega^1(\mathbb{R}^3 \setminus \{0\})$

then by Stokes' Thm $\int_{S^2} \omega = \int_{S^2} d\eta \underset{\pi}{=} \int_{\underset{\partial S^2}{\overset{\shortparallel}{\emptyset}}} \eta = 0$ ↯

$\Rightarrow$ ω not exact!

(2) Let $T_{r,R} = \{(x,y,z) \mid (\sqrt{x^2+y^2}-R)^2 + z^2 = r^2\}$

$0 < r < R$

a) Consider $\phi(\vartheta, \varphi) := ((R + r\cos\vartheta)\cos\varphi, (R + r\cos\vartheta)\sin\varphi, r\sin\vartheta)$

$\Rightarrow \quad \phi_1 = \phi|_{\underbrace{(0,2\pi)\times(0,2\pi)}_{V_1}}$

$\phi_2 = \phi|_{\underbrace{(-\pi,\pi)\times(0,2\pi)}_{V_2}}$

$\phi_3 = \phi|_{\underbrace{(0,2\pi)\times(-\pi,\pi)}_{V_3}}$

$\phi_4 = \phi|_{\underbrace{(-\pi,\pi)\times(-\pi,\pi)}_{V_4}}$

are injective, smooth maps s.t.

$\phi_1(V_1) \cup \phi_2(V_2) \cup \phi_3(V_3) \cup \phi_4(V_4) = T_{r,R}$

and $\quad \phi_j^{-1} \circ \phi_i = id \quad$ for $i,j \in \{1,2,3,4\}$

$\Rightarrow$ all coord. changes are smooth diffeos

$\Rightarrow \{U_i = \phi_i(V_i), \phi_i^{-1}, V_i\}_{i=1,2,3,4}$

is a smooth atlas of $T_{r,R}$

$\Rightarrow$ $T_{r,R}$ is a 2-dim'l manifold.

b) Let $\alpha = (\phi_i^{-1})^* d\vartheta$

$\beta = (\phi_i^{-1})^* d\varphi$

well-defined on entire $T_{1,R}$ independently from $i = 1,2,3,4$

Let $c(t) = \phi(t,0)$, $t \in [0,2\pi]$
$d(t) = \phi(0,t)$

These are closed curves on $T_{1,R}$ s.t.

$$\int_{[0,2\pi]} c^*\alpha = 2\pi = \int_{[0,2\pi]} d^*\beta$$

If $\alpha = df$ for $f \in C^\infty(T_{1,R})$

then $\int c^* df = 0$ because c is closed ↯

analogously for $\beta \neq dg$.

c) $\alpha \wedge \beta = (\phi_i^{-1})^* d\vartheta \wedge d\varphi$

$$\Rightarrow \int_{T_{1,R}} \alpha \wedge \beta = \int_{[0,2\pi]^2} d\vartheta \wedge d\varphi = 4\pi^2 \,//$$

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$$\omega_0 = dx^1 \wedge dy^1 + \ldots + dx^n \wedge dy^n$$

a) Let $X = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x^i} + g_i \frac{\partial}{\partial y^i}$, $f_i, g_i \in C^\infty(M)$

$$\Rightarrow \quad \iota_X \omega_0 = \sum_{i=1}^{n} f_i \, dy^i - g_i \, dx^i$$

The inverse of $\phi : X \to \iota_X \omega_0$

$$\text{is} \quad \phi^{-1}\left(\sum_{i=1}^{n} a_i \, dx^i + b_i \, dy^i \right)$$

$$= \sum_{i=1}^{n} b_i \frac{\partial}{\partial x^i} - a_i \frac{\partial}{\partial y^i}$$

i.e. $(f_i, g_i)_{i=1,\ldots,n} \Leftrightarrow (-g_i, f_i)_{i=1,\ldots,n}$ //

b) Let $\iota_{X_H} \omega_0 = dH = \sum_{i=1}^{n} \partial_{x^i} H \, dx^i + \partial_{y^i} H \, dy^i$

$$\Rightarrow \quad X_H = \sum_{i=1}^{n} \partial_{y^i} H \frac{\partial}{\partial x^i} - \partial_{x^i} H \frac{\partial}{\partial y^i}$$

Let $z = (x^1, \ldots, x^n, y^1, \ldots, y^n)$

then $\dot{z} = X_H \circ z \Leftrightarrow$ $\boxed{\dot{x}^i = \frac{\partial H}{\partial y^i} \text{ and } \dot{y}^i = -\frac{\partial H}{\partial x^i}}$ //

(c) Have for $X = \sum A_i \frac{\partial}{\partial x^i}$, $Y = \sum B_i \frac{\partial}{\partial x^i}$

$$[X, Y] = \sum_{ij} \left(A_i \frac{\partial B_j}{\partial x^i} - B_i \frac{\partial A_j}{\partial x^i} \right) \frac{\partial}{\partial x^j}$$

$$\Rightarrow [X_f, X_g] = \left[\sum_i \frac{\partial f}{\partial y^i}\frac{\partial}{\partial x^i} - \frac{\partial f}{\partial x^i}\frac{\partial}{\partial y^i}, \sum_j \frac{\partial g}{\partial y^j}\frac{\partial}{\partial x^j} - \frac{\partial g}{\partial x^j}\frac{\partial}{\partial y^j} \right]$$

$$= \sum_{ij} \left(\frac{\partial f}{\partial y^i} \frac{\partial^2 g}{\partial x^i \partial y^j} \frac{\partial}{\partial x^j} - \frac{\partial g}{\partial y^i} \frac{\partial^2 f}{\partial x^i \partial y^j} \frac{\partial}{\partial x^j} \right)$$

$$- \sum_{ij} \left(\frac{\partial f}{\partial x^i} \frac{\partial^2 g}{\partial y^i \partial y^j} \frac{\partial}{\partial x^j} - \frac{\partial g}{\partial x^i} \frac{\partial^2 f}{\partial x^i \partial y^j} \frac{\partial}{\partial y^j} \right)$$

$$- \sum_{ij} \left(\frac{\partial f}{\partial y^i} \frac{\partial^2 g}{\partial x^i \partial x^j} \frac{\partial}{\partial y^j} - \frac{\partial g}{\partial x^i} \frac{\partial^2 f}{\partial y^i \partial y^j} \frac{\partial}{\partial x^j} \right)$$

$$+ \sum_{ij} \left(\frac{\partial f}{\partial x^i} \frac{\partial^2 g}{\partial y^i \partial x^j} \frac{\partial}{\partial y^j} - \frac{\partial g}{\partial x^i} \frac{\partial^2 f}{\partial y^i \partial x^j} \frac{\partial}{\partial y^j} \right)$$

Suppose $\{f, g\} = A = A(f, g) \in C^\infty(U)$

then $X_A = \sum_i \frac{\partial A}{\partial y^i} \frac{\partial}{\partial x^i} - \frac{\partial A}{\partial x^i} \frac{\partial}{\partial y^i}$

$X_{\{f,g\}} = [X_f, X_g] \iff \frac{\partial A}{\partial y^j} = \sum_i \left(\frac{\partial f}{\partial y^i} \frac{\partial^2 g}{\partial x^i \partial y^j} - \frac{\partial g}{\partial y^i} \frac{\partial^2 f}{\partial x^i \partial y^j} - \frac{\partial f}{\partial x^i} \frac{\partial^2 g}{\partial y^i \partial y^j} + \frac{\partial g}{\partial x^i} \frac{\partial^2 f}{\partial y^i \partial y^j} \right)$

and $\frac{\partial A}{\partial x^j} = \sum_i \left(-\frac{\partial g}{\partial y^i} \frac{\partial^2 f}{\partial x^i \partial x^j} + \frac{\partial f}{\partial y^i} \frac{\partial^2 g}{\partial x^i \partial x^j} - \frac{\partial f}{\partial x^i} \frac{\partial^2 g}{\partial y^i \partial x^j} + \frac{\partial g}{\partial x^i} \frac{\partial^2 f}{\partial y^i \partial x^j} \right)$

(8)

$$\Leftrightarrow A = \sum_i \frac{\partial f}{\partial y^i}\frac{\partial g}{\partial x^i} - \frac{\partial f}{\partial x^i}\frac{\partial g}{\partial y^i} + \text{const}$$

(d) $\frac{d}{dt}\left(H \circ \phi_H^t\right) = X_H(H) = dH(X_H) = \omega(X_H, X_H) = 0$

(4) Consider coordinate transformation

$$q = (q^1, \dots, q^n) \to \bar{q} = (\bar{q}^1, \dots, \bar{q}^n)$$

$$\Rightarrow d\bar{q}^i = \sum_j \underbrace{\frac{\partial \bar{q}^i}{\partial q^j}}_{= A^i_j} dq^j = \sum_j A^i_j dq^j$$

Given $\alpha \in T_q^* Q$ have coord. representation

$$\alpha = \sum_i p_i dq^i = \sum_i \bar{p}_i d\bar{q}^i = \sum_{ij} \bar{p}_i A^i_j dq^j$$

$$\Rightarrow p_i = \sum_j \bar{p}_j A^j_i(q), \quad \bar{p}_j = \sum_i B^i_j(q) p_i$$

with $B = A^{-1}$

Hence $(q, p) \mapsto (\bar{q}, \bar{p} = B(q)p)$, $\frac{\partial \bar{q}}{\partial p} \equiv 0$

$$\Rightarrow \frac{\partial}{\partial p_i} = \sum_j \frac{\partial \bar{p}_j}{\partial p_i}\frac{\partial}{\partial \bar{p}_j} = \sum_j B^i_j \frac{\partial}{\partial \bar{p}_j}, \quad \frac{\partial}{\partial p} = B^T \frac{\partial}{\partial \bar{p}}$$

$$\Rightarrow p \cdot \frac{\partial}{\partial p} = (A\bar{p}) \cdot (A^{-1})^T \frac{\partial}{\partial \bar{p}} = \bar{p} \cdot \frac{\partial}{\partial \bar{p}}$$