Solution of Problem 4c on Series 2

Let $U \subset \mathbb{R}^n$ be an open subset. Any smooth map

$$T\colon U \to \bigotimes^2 (\mathbb{R}^n)^* \otimes \mathbb{R}^n, \quad p \mapsto T_p,$$

is called a 2-1-tensor field, also briefly a 2-1-tensor.
Note that for any vector fields $X, Y \in \mathcal{X}(U)$, we have that $T(X,Y)$ is again a vector field by

$$T(X,Y)(p) \;=\; T_p\big(X(p), Y(p)\big)\,.$$

So, every 2-1-tensor field T induces an $\mathbb{R}$-bilinear operation $T\colon \mathcal{X}(U) \times \mathcal{X}(U) \to \mathcal{X}(U)$. Moreover, we necessarily have for any functions $f, g \in C^\infty(U,\mathbb{R})$

$$T(fX, gY) \;=\; fgT(X,Y)\,, \tag{1}$$

because by the multilinearity of 2-1-tensors in $\bigotimes^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$ we have

$$T_p\big(f(p)X(p), g(p)Y(p)\big) \;=\; f(p)g(p)T_p\big(X(p), Y(p)\big)\,.$$

However, for any connection $\nabla\colon \mathcal{X}(U) \times \mathcal{X}(U) \to \mathcal{X}(U)$ we have

$$\nabla_{fX} gY \;=\; fg\nabla_X Y + fX(g)Y\,, \tag{2}$$

i.e.

$$(\nabla_{fX} gY)(p) \;=\; f(p)g(p)(\nabla_X Y)(p) + f(p)(X(g))(p)Y(p)\,.$$

So, the extra term $fX(g)Y$ prevents ∇ from being a 2-1-tensor.
If we now take two different connections ∇, ∇', however, we see from (2) for the expression $S(X,Y) := \nabla_X Y - \nabla'_X Y$ that

$$S(fX, gY) = fg(\nabla_X Y - \nabla'_X Y) \;=\; fgS(X,Y)\,.$$

So, the necessary condition (1) for the tensor property is fulfilled. (1) is in fact also a sufficient condition if S is $\mathbb{R}$-linear:
Namely, if we represent vector fields X, Y in local coordinates $(x^1, \dots, x^n)$ as

$$X = \sum_{i=1}^n a^i \frac{\partial}{\partial x^i} \qquad Y = \sum_{i=1}^n b^i \frac{\partial}{\partial x^i},$$

with coefficient functions $(a^i)_{i=1,\dots,n}$ and $(b^i)_{i=1,\dots,n}$, we see that

$$S(X,Y) \;=\; \sum_{i,j} a^i b^j S(\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^j})$$

so that $S(X,Y)$ at a point p depends only on the values of a^i and b^j at p itself. Hence, for any X, X' and Y, Y' with $X(p) = X'(p)$ and $Y(p) = Y'(p)$, we have $S(X,Y)(p) = S(X',Y')(p)$. Thus, S is in fact a 2-1-tensor field.

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Similarly, we compute for the torsion T_∇ of a connection ∇

$$
\begin{aligned}
T_\nabla(fX, gY) &= \nabla_{fX}(gY) - \nabla_{gY}(fX) - [fX, gY] \\
&= fg\nabla_X Y + fX(g)Y - gf\nabla_Y X - gY(f)X \\
&\quad - fgX \circ Y + gfY \circ X - fX(g)Y + gY(f)X \\
&= fg(\nabla_X Y - \nabla_Y X) - fg[X, Y] \\
&= fg\, T_\nabla(X, Y)\,.
\end{aligned}
$$

thus, T_∇ satisfies the same property (1) and hence is also a 2-1-tensor field.