Lecture 1. 16.OCT 2019. mathematical_physics
Leipzig Uni.

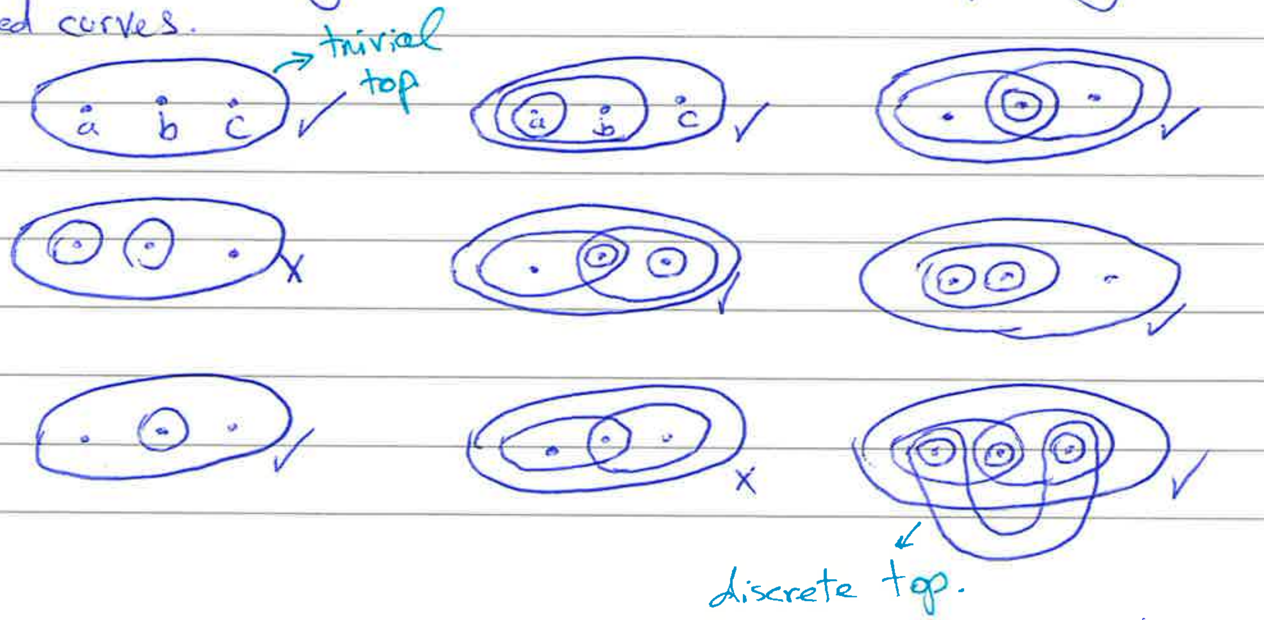
During the first decades of the 20th century there were different definitions proposed for a topological space and it took quite a while until they agreed on one which was the most suitable one:

DEF 1. A $\boxed{\text{Topology}}$ on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

① $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.
② the union of elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.
③ the intersection of elements of any finite " " " " " .

$(X, \mathcal{T})$ is called a $\boxed{\text{topological space}}$

EX 2. which of the following are top. sp. if $X = \{a, b, c\}$ and $\mathcal{T}$ is including $\emptyset$ and subsets of X corresponding to closed curves.



* A top. on X containing only ϕ and X is called <u>trivial top.</u> and a top. on X containing all the subsets of X is called <u>discrete top.</u>

In example <u>2</u> you see that even a 3-elements set has many different topologies, but not every collection of subsets of it is a top.

DEF 3. Let (X, τ) ~~be a~~ be a top. sp. and $U \in \mathcal{T}$, i.e. U is an open subset of X and let $x \in U$, Then U is called an $\boxed{\text{open neighborhood}}$ or just $\boxed{\text{neighborhood (nbhd)}}$ of x.

* The following thm is a way to check whether a subset is open or not.

Thm 4. (p.20 in lecture notes.) let (X, τ) be a top. sp.

$$U \subseteq X \text{ open} \iff \forall x \in U, \text{ there exists an open nbhd } U_x \text{ of } x \text{ s.t. } U_x \subset U$$

pf. ($\Rightarrow$) this is the trivial direction. let $U_x = U \quad \forall x \in U$.

($\Leftarrow$) let $U = \bigcup_{x \in U} U_x \rightsquigarrow$ this is a union of opens, so is open.

Ex 5. (especial case of proposition 19 in lecture notes.)

Let X be the real plane $\mathbb{R}^2$ and $\forall x_0 \in X$, $\varepsilon > 0$ define

$$B_\varepsilon(x_0) := \{x \in X \mid d(x, x_0) < \varepsilon\}$$

where $d(x, x_0)$ is the Eucledean distance of x and x_0.

i.e. $B_\varepsilon(x_0)$ is

this is an example of a metric sp. that you will study in your lecture.

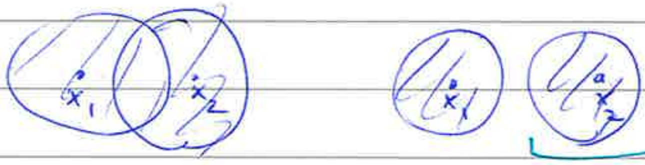
Now Let $\tau = \{U \subset X \mid \forall x \in U, \exists \varepsilon > 0 \text{ s.t. } B_\varepsilon(x) \subset U\}$

First of all let

$$\tau' = \{B_\varepsilon(x) \mid x \in X, \varepsilon > 0\}$$

is (X, τ') a top. sp.? why?

NO :



you see in your lecture that this space is hausdorff.

$B_\varepsilon(x_1) \cup B_\varepsilon(x_2)$ is not necessarily a ball.

show that τ is a top on X.

pf. we need to check 3 properties of DEF 1.

① $\phi \in \tau$ and $X \in \tau$ ✓

② WTS $\bigcup_{i \in I} U_i \in \tau$ if $U_i \in \tau$ $\forall i \in I$.

we use Thm 4. (Note that $\forall \varepsilon > 0, \forall x \in X$ $B_\varepsilon(x) \in \tau$)

used in ③ only.

$x \in \bigcup_{i \in I} u_i$, $\exists_i$ s.t. $x \in u_i \xrightarrow{u_i \in \mathcal{T}}$ $x \in u_i \subset \bigcup_{i \in I} u_i$

so $\bigcup_{i \in I} u_i$ is open.

③ $x \in \bigcap_{i=1}^{n} u_i \Rightarrow x \in u_i \quad \forall_{i=1,\dots,n}$ so

$\forall_{i=1,\dots,n}$ there exists $\varepsilon_i > 0$ s.t. $B_{\varepsilon_i}(x) \subset u_i$

let $\varepsilon = \min\{\varepsilon_i\} > 0$ so $B_{\varepsilon}(x) \subset u_i \quad \forall_i$

this exists and is positive bcz the finiteness.

$\Rightarrow B_{\varepsilon}(x) \subset \bigcap_{i=1}^{n} u_i$ so $\bigcap_{i=1}^{n} u_i \in \mathcal{T}$.

*This proof is true for a general metric space.

Sometimes it is more convenient to not present the set of all open sets and instead present a set which generates it.

DEF 6. If X is a set, a $\boxed{\text{basis}}$ for a top. on X is a collection $\mathcal{B}$ of subsets of X (called $\boxed{\text{basis elements}}$) s.t.

① $\forall x \in X$, $\exists B \in \mathcal{B}$ s.t. $x \in B$

② If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ then $\exists B_3 \in \mathcal{B}$ s.t. $x \in B_3 \subset B_1 \cap B_2$.

Let

$$\tau = \{U \subseteq X : \forall x \in U, \exists B \in \mathcal{B} \text{ s.t. } x \in B \subset U\}$$

then τ is called the $\boxed{\text{top. generated by } \mathcal{B}}$.

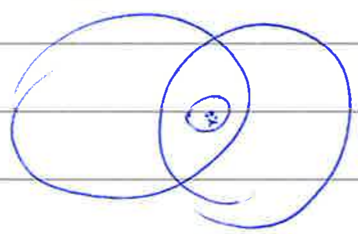
* you can check that τ is a top. on X. (check ①,②,③)

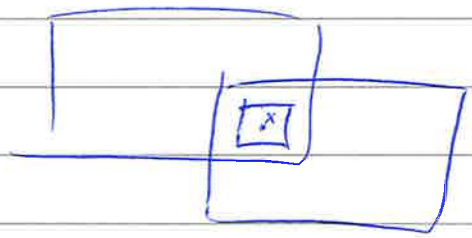
You can also equivalently define the top. gen. by $\mathcal{B}$ to be <u>the collection of all unions of elements of $\mathcal{B}$.</u>

* you can prove that this two top. are the same. (not trivial)

Ex 7. (examples of Basis).

In the plane let $\mathcal{B}_1$ be the collection of all circular regions, and $\mathcal{B}_2$ be the collection of all rectangular regions, check that both are basis of two top τ_1 and τ_2 on the plane. (check ①, ② in DEF 6)





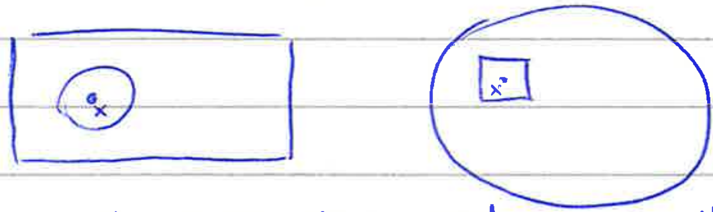
* Once you have a tool to practically check how two top. are equivalent you ~~see~~ can easily see that τ_1 and τ_2 are the same top!

Here is the tool:

Lemma 8: let (X,τ_1) and (X,τ_2) be two top. sp. τ_1 gen. by $\mathcal{B}_1$ and τ_2 gen by $\mathcal{B}_2$:

$$\tau_1 \subseteq \tau_2 \iff \forall x \in X,\ \forall B_1 \in \mathcal{B}_1 \text{ s.t. } x \in B_1,\ \exists B_2 \in \mathcal{B}_2 \text{ s.t. } x \in B_2 \subset B_1$$

Ex 9. τ_1 and τ_2 in Ex7 are the same top.



using lemma 8, this picture is the Proof :)!

~~One way to keep around~~

Example of ~~a~~ top. with basis on real line:

Ex 10. Let $X = \mathbb{R}$, $\mathcal{B}$ = collection of all open intervals in $\mathbb{R}$
i.e. the collection of all

$$(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$$

The top τ generated by $\mathcal{B}$ is called the $\boxed{\text{standard top}}$ on $\mathbb{R}$.

* Can you check here that why only finite intersection of opens are open and not infinite?

$$\longrightarrow \bigcap_{n=1}^{\infty} \left(\frac{-1}{n}, \frac{1}{n}\right) = \{0\} \longrightarrow \text{not open!}$$

now knowing a top. on a top. sp. X one can think of the subspaces of X, how they ~~top is defined~~ inherit the top?

NOT presented in class, will be sent to them via email.

(NO TIME)

DEF 11. Let (X, τ) be a top. sp. and $Y \subset X$. let

$$\tau_Y := \{ Y \cap U \mid U \in \tau \}$$

then (Y, τ_Y) is a top. sp. (check it.)

τ_Y is called the $\boxed{\text{subspace topology}}$

* Important: Not always the subspace top. on Y is the same as the top. on X restricted to Y. (you will soon see the example.)

- $a<a$ never holds
- $a<b$ and $b<c \Rightarrow a<c$
- $a<b$ or $b<a$.

Ex 12. let X be a set with simple order relation

Let $\mathcal{B}$ be the collection of

① (a,b) in X

② $[a_0, b)$ where $a_0 = \min X$ if exist.

③ $(a, b_0]$ ~ $b_0 = \max X$ ~ ~ .

let τ be gen. by $\mathcal{B}$, (X, τ) is a top. sp. τ is called

the $\boxed{\text{order topology}}$ on X.

* on $\mathbb{R}$ order top and ~~subspace~~ standard top. are the same bcz there is no min and max.

Now ~~Ex~~ Let $X = \mathbb{R}$ and $Y = [0,1) \cup \{2\}$

In subspace top. on Y, $\{2\}$ is open bcz $\{2\} = Y \cap \underbrace{(3/2, 5/2)}_{\text{open in } X}$

but let Y have order top. then $\{2\}$ is not open.

If $\{2\}$ is open, there has to be $(a, 2] \subset \{2\}$ for some $a \in Y$; $a < 2$.

but such interval does not exist!

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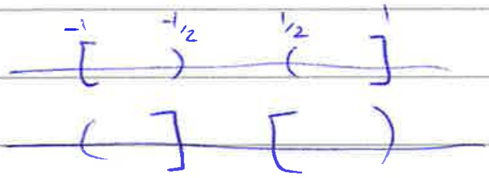
Ex 13. Consider $Y = [-1,1]$ as a subspace of $\mathbb{R}$. Which of the following sets are open in Y? Which are open in $\mathbb{R}$?

$A = \{x \mid \frac{1}{2} < |x| \leq 1\}$

$B = \{x \mid \frac{1}{2} \leq |x| < 1\}$

$C = \{x \mid 0 < |x| < 1 \text{ and } \frac{1}{x} \notin \mathbb{Z}_+\}$



A is not open in $\mathbb{R}$ but is open in Y bcz

$$A = Y \cap \underbrace{\left[(-\infty, \tfrac{-1}{2}) \cup (\tfrac{1}{2}, +\infty)\right]}_{\text{open in } \mathbb{R}}.$$

B not open in $\mathbb{R}$ and not open in Y. (there is no open nbhd of $\pm\frac{1}{2}$ in B.

$$C = \left(\bigcup_{n=1}^{\infty} \left(\frac{1}{n+1}, \frac{1}{n}\right)\right) \cup \left(\bigcup_{n=1}^{\infty} \left(\frac{-1}{n}, \frac{-1}{n+1}\right)\right) \quad \text{open in } \mathbb{R}.$$

also $C = C \cap Y$ ~~us~~ so open in Y as well.

Lemma 14. If $\mathcal{B}$ is a basis for the top on X, then $\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$ is basis for the subspace top on Y.

DEF 15 :

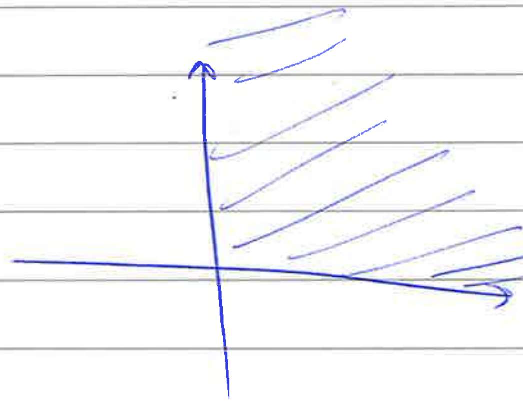
~~closed sets~~ A subset A of a top sp. on X is said to be $\boxed{\text{clsd}}$ if the set $X - A$ is open.

EX 16: examples: ① $[a,b] \subset \mathbb{R}$

$\mathbb{R} - [a,b] = (-\infty, a) \cup (b, +\infty)$

② $\{x \times y \mid x \geq 0 \text{ and } y \geq 0\}$

Bcz $= \mathbb{R}^2 - \left[(\mathbb{R} \times (-\infty, 0)) \cup (-\infty, 0) \times \mathbb{R} \right]$



③ now consider the subset $Y = [0,1] \cup (2,3) \subset \mathbb{R}$

this is not open in $\mathbb{R}$ bcz there is no nbhd $U \ni$ ~~of~~ 0 s.t. $U \subset Y$.

the complement is $Y^c = (-\infty, 0) \cup (1,2] \cup [3, +\infty)$.

this is also not open for the ~~same~~ similar reason.

so Y is not open and not closed.

④ let Y be the whole $\mathbb{R}$. the Y is both open and closed.

↳ there are non trivial examples in case someone ask!

→ So you see that there is a difference between a set being open or clsd and a door. :D

$[0,1] \cup \{2\}$ ↓ $\{2\}$ is open and closed in subspace top.

* a set can be open, closed, none, both :)

Thm 17.1: Let X be a top sp. then:

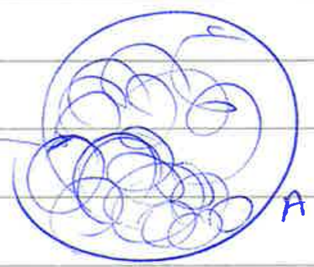
① $\emptyset$ and X are both closed.
② Arbitrary intersections of clsd sets are clsd.
③ finite Unions of clsd sets are clsd.

Thm ~~18~~ 17.2: let Y be a subspace of X. Then a set A is clsd in Y iff $A = X \cap C$ for some C closed in X.
(similar to the def. of open)

$\boxed{\text{Interior}}$ and $\boxed{\text{closure}}$: (DEF 19.)

Given a subset A of a top sp. X:

$$\boxed{\mathrm{Int}(A)} := \bigcup_{\substack{\text{st. } U \subset A \\ U \text{ open in } X}} U$$



this is clearly open.

closure of A $\to$ $\boxed{\overline{A}} := \bigcap_{\substack{A \subset C \\ C \text{ clsd in } X.}} C$ $\rightsquigarrow$ this is clearly clsd.

we have $\mathrm{Int}(A) \subset A \subset \overline{A}$.

If A is open $A = \mathrm{Int}(A)$ and
" A " clsd $A = \overline{A}$.

EX 20: $A = (0, \frac{1}{2}] \subset X = \mathbb{R}$
$\overline{A} = [0, \frac{1}{2}]$ $\quad$ $\mathrm{Int}(A) = (0, \frac{1}{2})$

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Thm 21: let A be a subset of the top sp. X

(a) $x \in \bar{A} \iff$ every open $U \subset X$ that $x \in U$ intersects A.

(b) If $\mathcal{B}$ is a basis for X, then $x \in \bar{A}$ $\iff$ every $B \subset \mathcal{B}$ containing x intersects A.

$X = \mathbb{R}$

EX22: $B = \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$ $\quad \bar{B} = \{0\} \cup B$

$\mathbb{Q}$ = rational numbers. $\quad \bar{\mathbb{Q}} = \mathbb{R}$

17.4

Thm 23: let Y be a subspace of X, let A be a subset of Y, let $\bar{A}$ denote the closure of A in X. Then the closure of A in Y equals $\bar{A} \cap Y$.

EX 24: $Y = (0,1] \subset X = \mathbb{R}$ (subspace) $\quad A = (0, \frac{1}{2})$

closure of A in $\mathbb{R} = \bar{A} = [0, \frac{1}{2}]$

" " " " $Y = [0, \frac{1}{2}] \cap (0,1] = (0, \frac{1}{2}]$

limit point: DEF 25:

let X be a top. sp. and A a subset of X. a point $x \in X$ is a $\boxed{\text{limit point}}$ of A if every nbhd of x (in X) intersects A in some point other than x itself.

we denote the set of all limit pts of A by A'.

EX. 26: $A = (0,1]$ $\quad A' = [0,1]$ every point of ~~is a limit point of A and they are all~~

$B = \{\frac{1}{n}, n \in \mathbb{Z}_+\}$ $\quad B' = \{0\}$

$\mathbb{Q}' = \mathbb{R}$ $\quad C = \{0\} \cup (1,2)$ $\quad C' = [1,2]$

Thm 26. (17.6) $A \subset X$: $\bar{A} = A \cup A'$

Corollary 27. (17.7) $A \subset X$, A clsd $\iff A' \subset A$
i.e. it includes all its limit pts.

DEF 28. A subset A of a top sp. X is said to be

$\boxed{\text{dense}}$ if $\bar{A} = X$.

EX 29: ▶ $\mathbb{Q} \subset \mathbb{R}$ is dense in $\mathbb{R}$ bcz $\overline{\mathbb{Q}} = \mathbb{R}$.
▶ $\mathbb{Q}^c \subset \mathbb{R}$ ~ ~ ~ ~ ~ $\overline{\mathbb{Q}^c} = \mathbb{R}$.
▶ $(0,1)$ is dense in $[0,1]$

▶ set $E = \{ x \mid 0 < |x| < 1 \text{ and } 1/x \notin \mathbb{Z}_+ \}$
is dense in $[-1,1]$
$\Rightarrow$ you have seen this set in Ex 13.

Thm 30 (Banach Fixed pt thm.) p38.

Let $F : X \to X$ be a map on a complete metric space

(X, d) s.t. $d(F(x), F(y)) \leq k \cdot d(x,y) \quad \forall x,y \in X$

where $k<1$ is fixed. Then there exists a unique $x_0 \in X$ s.t.

$F(x_0) = x_0$.

pf. Let $(x_n)_{n\in\mathbb{N}}$ be the sequence $x_{n+1} = F(x_n)$ in X. We will show that ① $\lim_{n\to\infty} x_n = p \in X$ (i.e. limit exists)

② $F(p) = p$

③ p is the unique fixed pt.

to prove ① we only need to show that $(x_n)_{n\in\mathbb{N}}$ is a cauchy sequence, then since X is complete, it will be converging to some $p \in X$.

to show that $(x_n)_{n\in\mathbb{N}}$ is a cauchy sequence, we have to prove that $\boxed{\forall \varepsilon > 0 \ \exists n_0 \in \mathbb{N} \text{ s.t. } \forall n > n_0 \text{ and } \forall k \in \mathbb{N} \quad d(x_{n+k}, x_n) < \varepsilon}$

$$d(x_{n+k}, x_n) = d(F(x_{n+k-1}), F(x_{n-1}))$$

$$< k\, d(x_{n+k-1}, x_{n-1}) \leq \cdots \leq k^n \underbrace{d(x_k, x_0)}$$

$$d(x_k, x_0) \leq \underbrace{d(x_k, x_{k-1})}_{\leq k^{k-1} d(x_1, x_0)} + \underbrace{d(x_{k-1}, x_{k-2})}_{\leq k^{k-2} d(x_1, x_0)} + \cdots + d(x_1, x_0)$$

$$\leq \underbrace{(k^{k-1} + k^{k-2} + \cdots + 1)}_{< \frac{1}{1-k}} \cdot d(x_1, x_0)$$

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So $d(x_{n+k}, x_n) \leq k^n \cdot \frac{1}{1-k} \cdot d(x_1, x_0)$

So $\forall \varepsilon > 0$ you need to choose n_0 big enough s.t.

$$k^{n_0} \cdot \frac{1}{1-k} d(x_1, x_0) < \varepsilon . \checkmark$$

② WTS $F(p) = p$

$$d(F(p), p) \leq \underbrace{d(F(p), \overset{F(x_{n-1})}{\overset{\shortparallel}{x_n}})}_{\leq k\, \underbrace{d(p, x_{n-1})}_{\to 0}} + \underbrace{d(p, x_n)}_{\to 0} \quad \forall n \in \mathbb{N}$$

$\Rightarrow d(F(p), p) = 0 \Rightarrow F(p) = p.$ /

③ let $F(q) = q$ for some $q \in X$

so $d(p, q) = d(F(p), F(q)) \leq k\, d(F(p), F(q))$

but $k < 1 \Rightarrow d(p, q) = 0 \Rightarrow p = q$.

DEF 31. A top. sp. X is $\boxed{\text{connected}}$ if it can not be written as the disjoint union of two nonempty open subsets. X is locally connected if $\forall x \in X$ and any open nbhd V of x, $\exists U$ open in X s.t. U is connected and $x \in U \subset V$.

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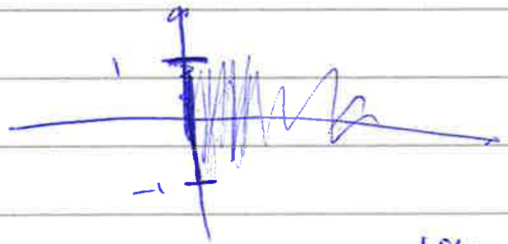
Important: Connectedness and locally connectedness are not related, X can be none of them, both or only one of them.

EX 32. let G be the graph of $f(x) = \sin\frac{1}{x}$ for $x \in (0,1]$ and I be the interval $\{0\} \times [-1,1]$.



let $X = I \cup G$ ~~with the~~ as a subspace of $\mathbb{R}^2$.

(let the top. on $\mathbb{R}^2$ be the one defined by Eucl. distance open balls, --)

X is not locally connected but is connected!

if X is not connected then $\exists A, B$ open non-empty subsets of X s.t. $X = A \cup B$ and $A \cap B = \emptyset$.

let $(x_0, \sin\frac{1}{x_0}) \in A$ for some $x_0 \in (0,1]$ then the entire graph G has to be in A. Let ~~[illegible]~~ s.~~[illegible]~~ $0 \in B$ but for any nbhd of 0 $\exists$ small enough $q > 0$ s.t. $(q, \sin\frac{1}{q})$ ~~[illegible]~~ is in that nbhd so $B \cap A \neq \emptyset$

so $0 \in A$ but then the whole $I \in A$ so $B = \emptyset$ ↯

$\bar{I}$ is not locally connected:

check a nbhd of 0.

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