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Problems in Mathematical Physics (12-PHY-MPMP1/2)
Sheet 3 (3 pages, Problems 3.1 – 3.4)

Problem 3.1

Let M be an n-dimensional manifold with a smooth atlas $\mathcal{A} = \{(\phi_i, U_i, V_i) \,|\, i \in I\}$ and $p \in M$. Recall the following definition of the tangent space of M at p: Two smooth curves $\gamma, \tilde{\gamma}\colon (-\epsilon, \epsilon) \to M$ with $\gamma(0) = \tilde{\gamma}(0) = p$ are considered equivalent if

$$\forall f \in C^\infty(M, \mathbb{R})\colon \qquad \frac{d}{dt}\bigg|_{t=0} (f \circ \gamma)(0) = \frac{d}{dt}\bigg|_{t=0} (f \circ \tilde{\gamma})(0)\,.$$

Then denote the equivalence class of γ by $[\gamma]_p = \dot{\gamma}(0)$ and the space of all equivalence classes as the tangent space

$$T_pM := \{\, [\gamma]_p \,|\, \gamma\colon (-\epsilon, \epsilon) \to M,\ \gamma(0) = p \,\}\,.$$

Secondly, let $I_p := \{\, i \in I \,|\, p \in U_i \}$ and denote the differential of the coordinate change by

$$\varphi_{ij} := D(\phi_i \circ \phi_j^{-1})(\phi_j(p)), \quad p \in U_i \cap U_j,\ i, j \in I_p\,.$$

In particular, we have $\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}$ for all $i, j, k \in I_p$. Then define

$$\widetilde{T}_pM := \{(v_i)_{i \in I_p} \in (\mathbb{R}^n)^{I_p} \,|\, v_i = \varphi_{ij}(v_j) \text{ f.a. } i, j \in I_p \,\}\,.$$

Thirdly, consider the vector space of derivations on the space of smooth functions on M,

$$\mathfrak{X}(M) := \{\, X \in \operatorname{Hom}(C^\infty(M, \mathbb{R}), C^\infty(M, \mathbb{R})) \,|\, X(fg) = fX(g) + gX(f),\ \text{f.a. } f, g \in C^\infty(M, \mathbb{R}) \,\}\,.$$

Consider two derivations $X, Y \in \mathfrak{X}(M)$ as equivalent with respect to $p \in M$ if and only if $X(f)(p) = X(g)(p)$ for all $f, g \in C^\infty(M, \mathbb{R})$, and denote the $\mathbb{R}$-vector space of equivalence classes $[X]_p$ by

$$\bar{T}_pM := \{[X]_p \,|\, X \in \mathfrak{X}(M) \,\}\,.$$

Show that all three tangent space definitions are equivalent in a canonical way, e.g. find isomorphisms

$$\psi_1\colon \bar{T}_pM \xrightarrow{\cong} T_pM,\ \psi_2\colon T_pM \xrightarrow{\cong} \widetilde{T}_pM,\ \psi_3\colon \widetilde{T}_pM \xrightarrow{\cong} \bar{T}_pM\,,$$

which do not depend on choices of bases or coordinate charts and such that $\psi_3 \circ \psi_2 \circ \psi_1 = \mathrm{id}$.
Hint: For ψ_1 choose an arbitrary coordinate chart with $p \in U_i$ and express the derivation in those coordinates. There, by a proposition from the class, it can be identified with a vector field which locally at p defines an integral curve. Show that the equivalence class of that curve on M is independent of the chosen coordinate chart. For ψ_3 work again in local coordinates and show that a given vector $v_i \in \mathbb{R}^n$ can be extended to a vector field such that its equivalence class $[X]_p$ at p is well-defined and independent of the coordinate chart. *4 pts.*

Problem 3.2

(a) Let φ and φ' be two inertial spacetime coordinate charts (fulfilling the axioms (G.N. 1...4)) and let $t \mapsto \underline{x}(t)$ be the coordinate trajectory of a pointlike object ("particle") with respect to φ. Assuming that the particle moves free of interaction, show that the coordinate trajectory $t' \mapsto \underline{x}'(t')$ with respect to φ' is given by a Galilei transformation, i.e.

$$t' = t + s\,, \qquad \underline{x}'(t') = \mathsf{D}\underline{x}(t) + \underline{v} \cdot t + \underline{x}_0$$

for all $t \in \mathbb{R}$, with suitable $\mathsf{D} \in O(3)$, $\underline{v}\,,\ \underline{x}_0 \in \mathbb{R}^3$, and $s \in \mathbb{R}$.

(You should derive this without using that the transition functions of inertial coordinate charts are given by Galilei transformations and prove it directly by going back to the definition of inertial coordinate charts. Thereby, you prove that the transition functions of inertial coordinate charts are exactly given by Galilei transformations.)

(b) Consider the case as in (a), where now the spacetime coordinate charts are assumed to fulfill (G.N. 1,2,4), but where the charts need not fulfill (G.N. 3), so they need not be inertial (but (G.N. 4) is extended to be valid for non-inertial charts fulfilling (G.N. 1,2)). Show that in this case, it holds that the coordinate trajectory $t' \mapsto \underline{x}'(t')$ is related to the coordinate trajectory $t \mapsto \underline{x}(t)$ by

$$t' = t + s\,, \qquad \underline{x}'(t') = \mathsf{D}(t)\underline{x}(t) + \underline{r}(t) \qquad (\star)$$

with functions $t \mapsto \mathsf{D}(t) \in O(3)$ and $t \mapsto \underline{r}(t) \in \mathbb{R}^3$ ($t \in \mathbb{R}$). Discuss the degree(s) of continuous differentiability of these maps. (And related with that, you may discuss why usually one restricts attention to $D(t) \in SO(3)$.)

(c) In the situation of (b), suppose that $t' = t$, and that φ is an inertial coordinate chart and φ' is a spacetime coordinate chart fulfilling (G.N. 1,2,4), so that $(\star)$ holds for the coordinate trajectories of any particle. Suppose that the equation of motion of the inertial coordinate trajectory is

$$m\frac{d^2}{dt^2}\underline{x}(t) = \underline{F}(t, \underline{x}(t), \dot{\underline{x}}(t))$$

with a (suitably smooth – at least continuous) function $\underline{F} : \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$. (We use the notation $\dot{\underline{x}}(t) = \frac{d}{dt}\underline{x}(t)$.)

Then in the other spacetime coordinates, the trajectory fulfills

$$m\frac{d^2}{dt^2}\underline{x}'(t) = \underline{F}'(t, \underline{x}'(t), \dot{\underline{x}}'(t))$$

Express $\underline{F}'(t, \underline{x}'(t), \dot{\underline{x}}'(t))$ using $\underline{F}$ as well as $\mathsf{D}(t)$, $\underline{r}(t)$ and suitable derivatives thereof, and $\underline{x}'(t)$ and $\dot{\underline{x}}'(t)$. (Interpret the terms that arise, like the Centrifugal "force" and the Coriolis "force" – however this is not actually part of the the problem as such.) *2 + 2 + 2 = 6 pts.*

Problem 3.3

Two Lagrange functions $\mathcal{L}_1$ and $\mathcal{L}_2$ for an n-particle system with holonomic constraints (f degrees of freedom) are said to *differ by a gauge transformation* if for every C^2 configuration space trajectory $t \mapsto q(t)$ the following holds:

$t \mapsto q(t)$ fulfills the Euler-Lagrange equations w.r.t. $\mathcal{L}_1$ if and only if $t \mapsto q(t)$ fulfills the Euler-Lagrange equations w.r.t. $\mathcal{L}_2$.

Show that, if $M : \mathbb{R} \times \mathsf{V} \to \mathbb{R}$ is C^3 (where $\mathsf{V} \subset \mathbb{R}^f$ is the (coordinatized) configuration space) and if

$$\mathcal{L}_2(t,q,u) = \mathcal{L}_1(t,q,u) + \sum_{k=1}^{f} \left(\frac{\partial}{\partial q_k} M(t,q) \right) u_k + \frac{\partial}{\partial t} M(t,q) \qquad (t \in \mathbb{R}\,,\ (q,u) \in \mathsf{V} \times \mathbb{R}^f \simeq T\mathsf{V})\,,$$

then $\mathcal{L}_1$ and $\mathcal{L}_2$ differ by a gauge transformation. *2 pts.*

Problem 3.4

Bead on the rotating wire. A bead of mass m (pointlike idealized) is bound to a wire (idealized as infinitesimally thin) on which it can move without friction. The wire is shaped into a planar circle of radius R. The wire is "standing", i.e. the x_3 axis passes through antipodal points (and the radial center) of the wire, and the wire rotates around the x_3 axis with constant angular velocity ω. The bead is subject to a homogeneous gravitional field along the x_3 axis in opposite direction, with a corresponding external gravitational potential $V(x_3) = mgx_3$ (where $g = 9.81\mathrm{m/s}^2$ is the gravitational acceleration).

(a) Determine suitable generalized coordinates and a Lagrange function for the motion of the bead.

(b) Derive the Euler-Lagrange equations for the motion of the bead.

(c) Special solutions to the Euler-Lagrange equations correspond to equilibrium points of the bead, i.e. to configuration space trajectories such that the bead is at rest in coordinates with respect to which the wire is at rest ("co-rotating coordinates"). Determine these equilibrium points.

4 pts.

Solutions due Wed 6 Nov in the exercise class.