UNIVERSITÄT LEIPZIG

Problem Sheet 13

Due Date: 29.06.2020, 12:00 UTC+2 (CEST)

Problem 1. [**2** pts] Use the definition of the Riemann integral to compute $\int_{-1}^{1} x^2 dx$.

Problem 2. [**2+2+2+3** pts] Compute the following integrals

$$\int_0^2 |1-x|dx, \qquad \int_0^{2\pi} x^2 cosx dx, \qquad \int_0^{\ln 2} \sqrt{e^x - 1}dx, \qquad \int_2^3 \frac{1}{6^x+1}dx$$

Problem 3. [**2+2+2+3** pts]
Compute the following integrals

$$\int_0^1 \frac{1}{\sqrt{1-x}}dx, \qquad \int_{-10}^{10} \frac{dx}{2x^2+9x-5}dx, \qquad \int_0^\infty x^2 e^{-x} dx.$$

Does the integral $\int_0^\infty x^p e^{-x} dx$ converge for any $p \geq 0$? Does it converge for any $p < 0$?

Problem 4. [**2+2+3** pts] Find an antiderivative (including its domain) of

$$x^{10} lnx, \qquad sin^5 x cosx, \qquad sin^4 x$$

Problem 5. [**4** pts] Compute length of the curve formed by points $x \in \mathbb{R}^2$ as follows: distance of x from the origin equals $1 + cos\theta$, where $\theta \in [0, 2\pi]$ is the angle between the axis OX and the line linking x and the origin.

Problem 6. [**3** pts] State and prove a generalisation of the Jensen inequality from $[0,1] \subset \mathbb{R}$ (that appears in lecture notes) to any $[a,b] \subset \mathbb{R}$ with a, b finite real numbers.

Problem 7. [**4** pts] Let $a > b$ be two real numbers. Let $f : \mathbb{R} \to \mathbb{R}$ be such that: $f(a) = f(b) = 0$, derivative f' is continuous on $[a,b]$, and

$$\int_a^b f^2(x)dx = 1$$

Prove that

$$\int_a^b x f(x) f'(x) dx = -\frac{1}{2} \qquad \text{and} \qquad \left(\int_a^b (f'(x))^2 dx\right)\left(\int_a^b x^2 f^2(x) dx\right) > \frac{1}{4}.$$

Problem 8. [* **5** pts] Fix a function f that is Riemann integrable on $[a,b]$, a, b finite real numbers. Fix any $\epsilon > 0$. Prove that there exists a polynomial $g : [a,b] \to \mathbb{R}$ such that

$$\int_a^b |f(x) - g(x)|^2 dx \leq \epsilon.$$