Problem sheet 11.

Solution 2:

$S: \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \qquad S(x, y, z) = (x-y,\ x+3y+z,\ 2z)$

$$S = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\det \underbrace{(S - \lambda I)}_{A} = \det \begin{bmatrix} 1-\lambda & -1 & 0 \\ 1 & 3-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix} = -\lambda^3 + 6\lambda^2 - 12\lambda + 8$$
$$= (-\lambda + 2)^3$$

The only eigenvalue is $\lambda = 2$ and it has multiplicity $\boxed{3}$.

To find the corresponding eigenvectors

solve $Av = 0$ after substituting $\lambda = 2$ in A:

let $v = (v_1, v_2, v_3) \in \mathbb{R}^3$

$$\begin{bmatrix} -1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0 \longrightarrow \begin{cases} -v_1 - v_2 = 0 \longrightarrow v_1 = -v_2 \\ v_1 + v_2 + v_3 = 0 \end{cases}$$

The solution space is $\{(t, -t, 0) \in \mathbb{R}^3 : t \in \mathbb{R}\}$

The normalized basis for this space is

$v = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$

$\Longrightarrow \lambda = 2$ and $v = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ is the only eigenpair.

$S: \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \quad S(x, y, z) = (4x+y, 3x+2y, 7x-7y+5z)$

$$S = \begin{bmatrix} 4 & 1 & 0 \\ 3 & 2 & 0 \\ 7 & -7 & 5 \end{bmatrix}. \quad \text{Set } A := S - \lambda I$$

$$\det A = \det \begin{bmatrix} 4-\lambda & 1 & 0 \\ 3 & 2-\lambda & 0 \\ 7 & -7 & 5-\lambda \end{bmatrix} = -\lambda^3 + 11\lambda^2 - 35\lambda + 25$$

$$= -(\lambda-5)^2(\lambda-1)$$

$\Rightarrow \lambda_1 = 5$ and $\lambda_2 = 1$ are two eigenvalues.

The corresponding eigenvectors:

$$\lambda_1 = 5 : \; Av = 0 \rightarrow \begin{bmatrix} -1 & 1 & 0 \\ 3 & -3 & 0 \\ 7 & -7 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$X = \{ (v_1, v_2, v_3) \in \mathbb{R}^3 \text{ s.t. } v_1 = v_2 \text{ and } v_3 \in \mathbb{R} \}$

$$\left.\begin{aligned} v &= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \\ w &= (0, 0, 1) \end{aligned}\right\} \rightarrow \text{these are the basis for } X.$$

$$\lambda_2 = 1 : \; Au = 0 \rightarrow \begin{bmatrix} 3 & 1 & 0 \\ 3 & 1 & 0 \\ 7 & -7 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} 3u_1 + u_2 = 0 \\ 7u_1 - 7u_2 + 4u_3 = 0 \end{cases}$$

$$Y = \left\{ \left(u_1, -3u_1, \frac{1}{4}\left(\overbrace{-7u_1 + 7(-3u_1)}^{-7u_1}\right)\right) \in \mathbb{R}^3 \text{ s.t. } u_1 \in \mathbb{R} \right\}$$

$u = \left(\frac{-1}{\sqrt{59}}, \frac{3}{\sqrt{59}}, \frac{7}{\sqrt{59}}\right)$ is an orthonormal basis for Y.

(λ_1, v), (λ_1, w), (λ_2, u) are three eigenpairs of S.

Solution 4: $S: \mathbb{R}^3 \to \mathbb{R}^3$

$$S = \begin{bmatrix} 1 & \sqrt{3} & 1 \\ \sqrt{3} & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

eigenpairs of S: (to see how we find them check solution $\underline{2}$.

$\lambda_1 = 3 \to v_1 = \left(\sqrt{\frac{3}{7}}, \frac{1}{\sqrt{7}}, \sqrt{3/7}\right)$

$\lambda_2 = -\sqrt{2} \to v_2 = \left(-\frac{2+\sqrt{2}}{\sqrt{2(8+5\sqrt{2})}}, \frac{\sqrt{3}+\sqrt{6}}{\sqrt{2(8+5\sqrt{2})}}, \frac{1}{\sqrt{2(8+5\sqrt{2})}}\right)$

$\lambda_3 = \sqrt{2} \to v_3 = \left(\frac{-2+\sqrt{2}}{\sqrt{2(8-5\sqrt{2})}}, \frac{\sqrt{3}-\sqrt{6}}{\sqrt{2(8-5\sqrt{2}}}, \frac{1}{\sqrt{2(8-5\sqrt{2})}}\right)$

Let $Q = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix}$ i.e. the first column is v_1

the second ~ ~ v_2

~ third ~ ~ v_3

spectral decomposition gives:

$$S = Q^T D Q \quad \text{where} \quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -\sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}.$$

Now $$S^{101} = \underbrace{(Q^T D \overbrace{Q)(Q^T}^{\mathbb{I}} D Q) \cdots (Q^T D Q)}_{101 \text{ times}}$$

$$= Q^T D^{101} Q$$

$$= Q^T \begin{bmatrix} 3^{101} & 0 & 0 \\ 0 & -\sqrt{2}^{101} & 0 \\ 0 & 0 & \sqrt{2}^{101} \end{bmatrix} Q$$

exp is an analytic function therefore,

$$\exp(S) = Q^T \exp(D) Q$$

$$= Q^T \begin{bmatrix} \exp(3) & 0 & 0 \\ 0 & \exp(-\sqrt{2}) & 0 \\ 0 & 0 & \exp(\sqrt{2}) \end{bmatrix} Q.$$

Furthermore,
$$S^{1/2} = Q^T \sqrt{D}^{\mathbb{C}} Q$$

$$\sqrt{D}^{\mathbb{C}} = \begin{bmatrix} \pm\sqrt{3} & 0 & 0 \\ 0 & \pm i(2)^{1/4} & 0 \\ 0 & 0 & \pm 2^{1/4} \end{bmatrix}$$

D has $2^3 = 8$ complex square roots and as a result S has 8 square roots.

Problem 6.

Form the extended matrix $(A \vdots b)$:

First we check that for which values of s,t, $(A \vdots b)$ has rank 4, because then we know that for those values there is no solutions. (here both row and column operators are allowed).

(This is my first try and not the shortest way :))

$$\begin{array}{c|cccc|}
 & c_1 & c_2 & c_3 & c_4 \\
r_1 & 3 & 1 & 3 & 2 \\
r_2 & 2 & -2 & -1 & 0 \\
r_3 & 4 & 4 & 7 & t \\
r_4 & 5 & s & 11 & 0
\end{array}
\xrightarrow{c_3 = c_3 - c_1}
\begin{bmatrix} 3 & 1 & 0 & 2 \\ 2 & -2 & -3 & 0 \\ 4 & 4 & 3 & t \\ 5 & s & 6 & 0 \end{bmatrix}$$

$$\downarrow r_3 = r_3 + r_2$$

$$\begin{bmatrix} 3 & 1 & 0 & 2 \\ 2 & -2 & -3 & 0 \\ 0 & 0 & 0 & t-4 \\ 5 & s & 6 & 0 \end{bmatrix}
\xleftarrow{r_3 = r_3 - 2r_1}
\begin{bmatrix} 3 & 1 & 0 & 2 \\ 2 & -2 & -3 & 0 \\ 6 & 2 & 0 & t \\ 5 & s & 6 & 0 \end{bmatrix}$$

if $t \neq 4$ $\downarrow$

$$\begin{bmatrix} 3 & 1 & 0 & 2 \\ 2 & -2 & -3 & 0 \\ 0 & 0 & 0 & 1 \\ 5 & s & 6 & 0 \end{bmatrix}
\xrightarrow{r_1 = r_1 - 2r_2}
\begin{bmatrix} 3 & 1 & 0 & 0 \\ 2 & -2 & -3 & 0 \\ 0 & 0 & 0 & 1 \\ 5 & s & 6 & 0 \end{bmatrix}$$

$$\downarrow r_2 = r_2 + 2r_1$$

$$\begin{bmatrix} 1 & \frac{1}{3} & 0 & 0 \\ 0 & -\frac{8}{3}\Big/{-\frac{8}{3}} & -3\Big/{-\frac{8}{3}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & s-\frac{5}{3} & 6 & 0 \end{bmatrix}
\xleftarrow[r_4 = r_4 - 8r_1]{r_2 = r_2 - 8r_1}
\begin{bmatrix} 3/3 & 1/3 & 0 & 0 \\ 8 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 \\ 5 & s & 6 & 0 \end{bmatrix}$$

$$\downarrow r_4 = r_4 - \left(s - \frac{5}{3}\right) r_2$$

$$\begin{bmatrix} 1 & \frac{1}{3} & 0 & 0 \\ 0 & -\frac{8}{3} & -3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & s-7 & 0 & 0 \end{bmatrix}$$

if $s \neq 7$ $\longrightarrow$ we clearly have a full-rank matrix. i.e. we have four lin. indep. rows and ~ ~ ~ columns.

For the given SOLE to have solutions it should either have $s=7$ or $t=4$. we consider $\underline{3}$ cases:

- Set $s=7$ and $t=4$. The RREF of $(A \,\vdots\, b)$ after the substitution for s and t is:

$$\begin{bmatrix} 1 & 0 & 5/8 & 0 \\ 0 & 1 & 9/8 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

?

– In this case system has no solution.

- Set $t=4$ and $s \neq 7$. The RREF of $(A \,\vdots\, b)$ after this substitution is:

$$\begin{bmatrix} 1 & 0 & 0 & \frac{2(-22+s)}{9(-7+s)} \\ 0 & 1 & 0 & -\frac{6}{-7+s} \\ 0 & 0 & 1 & \frac{4(5+s)}{9(-7+s)} \end{bmatrix}$$

Therefore the system has the unique sol. $x = A^{-1}b$.

- Set $s=7$ and $t\neq 4$: The RREF of $(A \mid b)$

$$\begin{bmatrix} 1 & 0 & \frac{5}{8} & 0 \\ 0 & 1 & \frac{9}{8} & 0 \\ 0 & 0 & 0 & t-4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since $t \neq 4$

Therefore in this case the system has no solutions.

(1) a) $\varphi : \mathbb{R}^3 \to \mathbb{R}^4$; $\varphi(x,y,z) = (4x+3y+5z,\ x+2y+z,\ 2x-y+3z,\ 6x+7y+7z)$

Kernel of the linear map is defined as:

$$\mathrm{Ker}\,\varphi := \left\{ v \in \mathbb{R}^3 : \varphi(v) = 0 \in \mathbb{R}^4 \right\}$$

$$\varphi(v) = 0 \equiv \underbrace{\begin{bmatrix} 4 & 3 & 5 \\ 1 & 2 & 1 \\ 2 & -1 & 3 \\ 6 & 7 & 7 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Next we solve the SOLE by RR and find solutions:

$$Ax = 0 \sim \begin{array}{c} \begin{array}{ccc} w_1 & w_2 & w_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 7/5 \\ 0 & 1 & -1/5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \phi \end{bmatrix}$$

Now we read-off the system again:

$$\begin{cases} x + \frac{7}{5} z = 0 \\ y - \frac{1}{5} z = 0 \end{cases} \qquad \text{Take } z = c \in \mathbb{R} \text{ as free variable}$$

$$\Rightarrow \vec{x} = c \begin{pmatrix} -7 \\ 1 \\ 5 \end{pmatrix} \quad \Rightarrow \quad e_1 = \begin{pmatrix} -7 \\ 1 \\ 5 \end{pmatrix} \text{ is basis for } \mathrm{Ker}\,\varphi$$

Since columns of the matrix A span the $\mathrm{im}\,\varphi$ the dimension of $\mathrm{im}\,\varphi$ is # of independent column vectors: $\Rightarrow \dim(\mathrm{im}\,\varphi) = 2$

and $w_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $w_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ are basis for $\mathrm{im}\,\varphi$.

Note: rank-nullity formula is satisfied:

$$\mathrm{rank}\,\varphi + \mathrm{nullity}\,\varphi = \dim \mathbb{R}^3 = 3$$

(1)(b) : $\varphi : \mathbb{R}^4 \to \mathbb{R}^3$

$$\varphi(v)=0 \equiv \underbrace{\begin{bmatrix} 5 & 6 & 4 & 7 \\ 1 & 3 & 2 & 4 \\ 7 & 3 & 2 & 2 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$Ax = 0 \sim \begin{array}{cccc} w_1 & w_2 & w_3 & w_4 \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & -1/3 \\ 0 & 1 & 2/3 & 13/9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x_2 + \frac{2}{3} x_3 + \frac{13}{9} x_4 = 0 \\ x_1 - \frac{1}{3} x_4 = 0 \Rightarrow 3x_1 = x_4 \end{cases}$$

Now take $x_1 = t \in \mathbb{R}$ as free variable and $x_2 = s \in \mathbb{R}$ as well. $(t \neq 0 \wedge s \neq 0)$

$$\Rightarrow \vec{x} = \begin{pmatrix} t \\ s \\ \frac{3s+13t}{2} \\ 3t \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \\ 13/2 \\ 3 \end{pmatrix} + s \begin{pmatrix} 0 \\ 1 \\ 3/2 \\ 0 \end{pmatrix}$$

since t, s are arbitrary $2t$ and $2s$ are as well.

$$\Rightarrow \vec{x} = t \underbrace{\begin{pmatrix} 2 \\ 0 \\ 13 \\ 6 \end{pmatrix}}_{v_1} + s \underbrace{\begin{pmatrix} 0 \\ 2 \\ 3 \\ 0 \end{pmatrix}}_{v_2}$$

$\Rightarrow$ Basis for $\operatorname{Ker}\varphi$ are v_1, v_2. $\Rightarrow \dim(\operatorname{Ker}\varphi) = 2$

and w_1, w_2 are basis for $\operatorname{im}\varphi$.

$$\underbrace{2}_{\dim(\operatorname{Ker}\varphi)} + \underbrace{2}_{\dim(\operatorname{im}\varphi)} = 4 \longrightarrow \dim(\mathbb{R}^4)$$

3. Since $S: V \to V$ is defined by describing its action on an arbitrary basis $\{a^1, \ldots, a^d\}$ we cannot write down the matrix representation of S immediately. Therefore, we have to use the very important fact that the principle invariants of a Tensor and Hence its characterist Polynomial are indepentent of matrix representation of the Tensor in question. The change of basis Transformation is a Similarity Transformation and we need to find a matrix Similar to matrix representation of S.

$$[S] = A\,C\,A^{-1}$$

You were given a Hint that this matrix, C, is what we need and is the matrix rep. of a map $\varphi: \mathbb{R}^d \to \mathbb{R}^d$ in the canonical basis.

$$C = \begin{bmatrix} 0 & 0 & \cdots & & \alpha_1 \\ 1 & 0 & \cdots & & \alpha_2 \\ 0 & 1 & 0 & \cdots & \alpha_3 \\ \vdots & 0 & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & \alpha_d \end{bmatrix}$$

To find the change of Basis matrix Notice that We can write:

$$\begin{cases} a^1 = a_1^1 e_1 + a_2^1 e_2 + \cdots + a_d^1 e_d \\ a^2 = a_2^2 e_1 + a_2^2 e_2 + \cdots + a_d^2 e_d \\ \vdots \\ a^d = a_1^d e_1 + \cdots\cdots + a_d^d e_d \end{cases}$$

$$\Rightarrow Q_{ae} = [a^1, a^2, \ldots, a^d]$$
$$= [a^1, Sa^1, \ldots, S_a^{d-1}]$$

this is the matrix A given in the Hints.
$(A \leftrightarrow Q_{ae})$

Now we need to show this Change of basis matrix is invertible. But, that is easy! since $\{a^1, \ldots, a^d\}$ is basis, we know $a^1, \ldots, a^d$ are all linearly independent and Hence rank of A is full rank and therefore A is invertible.

Now all it has to be done is to find the Characteris polynomial of C and before that

Show that $S \overset{!}{=} ACA^{-1} \Leftrightarrow SA = AC$

post-multiply with A

i) $SA = S\left[a^1, a^2, \dots, a^d\right]$

$= \left[Sa^1, Sa^2, \dots, Sa^d\right]$

$= \left[a^2, a^3, \dots, \alpha_i a^i\right]$ (by def of S)

ii) $AC = A\begin{bmatrix} 0 & 0 & \cdots & \alpha_1 \\ 1 & 0 & \cdots & \alpha_2 \\ 0 & 1 & & \vdots \\ \vdots & 0 & \ddots & \\ 0 & \vdots & 1 & \alpha_d \end{bmatrix}$

$= \left[a^1, a^2, \dots, a^d\right]\begin{bmatrix} 0 & 0 & \cdots & \alpha_1 \\ 1 & 0 & & \alpha_2 \\ 0 & 1 & & \alpha_3 \\ \vdots & 0 & & \vdots \\ 0 & 0 & & \alpha_d \end{bmatrix}$

$= \left[a^2, a^3, \dots, \alpha_i a^i\right]$

And characteristic polynomial of C is found by using mathematical induction on dimension of matrix.

1) for $d=1$ we have the base case:

$$C = [\alpha_1].$$

$$\det(\omega \mathbb{1} - C) = |\omega - \alpha_1| = \omega - \alpha_1$$

2) for $d=2$ $\qquad C = \begin{bmatrix} 0 & \alpha_1 \\ 1 & \alpha_2 \end{bmatrix}$

$$\det(\omega \mathbb{1} - C) = \begin{vmatrix} \omega & \alpha_1 \\ 1 & \omega - \alpha_2 \end{vmatrix} = \omega(\omega - \alpha_2) - \alpha_1$$
$$= \omega^2 - \alpha_2 \omega - \alpha_1$$

3) for $d=3$ $\qquad C = \begin{bmatrix} 0 & 0 & \alpha_1 \\ 1 & 0 & \alpha_2 \\ 0 & 1 & \alpha_3 \end{bmatrix}$

$$\det(\omega \mathbb{1} - C) = \begin{vmatrix} \omega & 0 & \alpha_1 \\ 1 & \omega & \alpha_2 \\ 0 & 1 & \omega - \alpha_3 \end{vmatrix} = \omega \begin{vmatrix} \omega & \alpha_2 \\ 1 & \omega - \alpha_3 \end{vmatrix} - \alpha_1 \begin{vmatrix} 1 & \omega \\ 0 & 1 \end{vmatrix}$$

$$= \omega \cdot \{ \omega(\omega - \alpha_3) - \alpha_2 \} - \alpha_1$$

$$= \omega \cdot \{ \omega^2 - \alpha_3 \omega - \alpha_2 \} - \alpha_1 = \omega^3 - \alpha_3 \omega^2 - \alpha_2 \omega - \alpha_1$$

Now you see the pattern and prove it for all $d \geqslant 1$.

Note: Here using $\det(\omega \mathbb{1} - C)$ instead of $\det(C - \omega \mathbb{1})$ makes no difference for the purpose of finding eigenvalues. only difference is $(-1)^n$, i.e a sign!

$$C = \begin{bmatrix} 0 & 0 & 0 & \cdots & \alpha_1 \\ 1 & 0 & 0 & \cdots & \alpha_2 \\ 0 & 1 & 0 & \cdots & \alpha_3 \\ \vdots & 0 & 1 & & \\ & \vdots & 0 & & \\ 0 & 0 & 0 & \cdots & \alpha_d \end{bmatrix}$$

to find characteristic polynomial form $\det([C]-\omega[\mathbb{1}])$

$$\begin{vmatrix} -\omega & 0 & 0 & \cdots & \alpha_1 \\ 1 & -\omega & 0 & \cdots & \alpha_2 \\ 0 & 1 & -\omega & \ddots & \alpha_3 \\ 0 & 0 & 1 & \ddots & \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots 1 & \alpha_d-\omega \end{vmatrix} \underset{\left(\det = \frac{\det}{\omega}\right)}{\overset{R_2 \to \omega R_2}{=}} \begin{vmatrix} -\omega & 0 & 0 & \cdots & \alpha_1 \\ \omega & -\omega^2 & 0 & \cdots & \alpha_2\omega \\ 0 & 1 & -\omega & \cdots & \alpha_3 \\ 0 & 0 & & & \vdots \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & & 1 & \alpha_d-\omega \end{vmatrix}$$

$$\overset{R_1+R_2 \to R_2}{=} \begin{vmatrix} -\omega & 0 & 0 & \cdots & \alpha_1 \\ 0 & -\omega^2 & 0 & \cdots & \alpha_1+\alpha_2\omega \\ 0 & 1 & -\omega & \cdots & \alpha_3 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & 1 & \alpha_d-\omega \end{vmatrix}$$

expand C_1

$$= -\omega \cdot \begin{vmatrix} -\omega^2 & 0 & 0 & \cdots & \alpha_1+\alpha_2\omega \\ 1 & -\omega & 0 & \cdots & \alpha_3 \\ 0 & 1 & -\omega & & \alpha_4 \\ \vdots & 0 & & \ddots & \vdots \\ 0 & 0 & & 1 & \alpha_d-\omega \end{vmatrix} \underset{\left(\det = \frac{\det}{\omega^2}\right)}{\overset{R_2 \to \omega^2 R_2}{=}} \cdots$$

by doing the same Algorithm for $(d-2)$ steps

We reach the following

$$= (-\omega)(-\omega^2)(-\omega^3)\cdots(-\omega^{d-2}) \begin{vmatrix} -\omega^{d-1} & \alpha_1+\omega\alpha_2+\cdots+\alpha_{d-1}\omega^{d-2} \\ 1 & \alpha_d-\omega \end{vmatrix}$$

$R_2 \to \omega^{d-1} R_2$

$$\overset{\downarrow}{=} (-\omega)(-\omega^2)\cdots(-\omega^{d-2}) \begin{vmatrix} -\omega^{d-1} & \sum_{i=1}^{d-1} \alpha_i \omega^{i-1} \\ \omega^{d-1} & \omega^{d-1}(\alpha_d-\omega) \end{vmatrix}$$

$\left(\det = \dfrac{\det}{\omega^{d-1}}\right)$

$R_1+R_2 \to R_2$

$$= (-\omega)\cdots(-\omega^{d-2}) \begin{vmatrix} -\omega^{d-1} & \sum_{i=1}^{d-1} \alpha_i \omega^{i-1} \\ 0 & \sum_{i=1}^{d} \alpha_i \omega^{i-1} - \omega^d \end{vmatrix}$$

$$= (-\omega)\cdots(-\omega^{d-1}) \left[\left(\sum_{i=1}^{d} \alpha_i \omega^{i-1}\right) - \omega^d\right]$$

But now we have to devide the result for each step we multiplied R_2 by $\omega, \omega^2, \cdots$.

$$\Rightarrow \det([C] - \omega[1]) = (-1)^{d-1} \left[\left(\sum_{i=1}^{d} \alpha_i \omega^{i-1}\right) - \omega^d\right]$$

$$\Rightarrow \det \sim \; = \omega^d - \alpha_d \omega^{d-1} - \alpha_{d-1} \omega^{d-2} - \cdots - \alpha_2 \omega - \alpha_1 .$$

5. $S: \mathbb{R}^3 \to \mathbb{R}^3$;

Q is orthogonal $\overset{def}{\Longrightarrow}$ $QQ^T = Q^TQ = \mathbb{1}$

$\forall S$ $\quad I_1(S) = tr(S) \qquad I_2(S) = \frac{1}{2}\left[tr^2(S) - tr(S^2)\right]$

$I_3 = -\det S$

$\forall T, S \quad tr(ST) = tr(TS)$

$\Rightarrow$ i) $I_1(QSQ^T) = tr(QSQ^T) \overset{\downarrow}{=} tr(Q^TQS) \underset{\underset{QQ^T = \mathbb{1}}{\uparrow}}{=} tr(S)$

ii) $I_2(QSQ^T) = \frac{1}{2}\left[tr^2(QSQ^T) - tr\left((QSQ^T)(QSQ^T)\right)\right]$

$\underset{\underset{(i)}{\uparrow}}{=} \frac{1}{2}\left[tr^2(S) - tr(QS^2Q^T)\right] \underset{\underset{(i)\text{ applied to } S^2}{\uparrow}}{=} \frac{1}{2}\left[tr^2(S) - tr(S^2)\right]$

iii) $I_3 = -\det(QSQ^T) \underset{\underset{\forall S,T \quad \det(ST) = \det(TS)}{\uparrow}}{=} -\det(Q^TQS) = -\det(S)$

7. $T: V \to W$

$$\begin{cases} V = \mathrm{Span}(\{f^1, f^2, f^3\}) \\ W = \mathrm{Span}(\{g^1, g^2\}) \end{cases}$$

matrix of T in new basis

$$M_T^{fg} = \begin{bmatrix} 0 & -1 & 2 \\ 3 & -4 & 5 \end{bmatrix} \quad ; \quad M_T' := M_T^{f'g'} = ?$$

$$M_T' = Q_{g'g} \, M_T^{fg} \, Q_{ff'}$$

$$\begin{cases} \text{where } Q_{g'g} \text{ is matrix of change-of-basis } g' \to g. \\ \text{and } Q_{ff'} \text{ 〃 〃 〃 〃 } f \to f'. \end{cases}$$

Change of basis for $f \to f'$ is given as

$$\begin{cases} f_1' = f_1 \\ f_2' = f_1 + f_2 \\ f_3' = f_1 + f_2 - f_3 \end{cases} \Rightarrow Q_{ff'} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\begin{cases} g_1' = g_1 \\ g_2' = g_1 - g_2 \end{cases} \Rightarrow Q_{gg'} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

And $Q_{g'g} = Q_{gg'}^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$

$$\Rightarrow M_T' = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 2 \\ 3 & -4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$