Preparatory Exercises for the Problem Sheet 1

SS2020 - Analysis 2 - University of Leipzig

Mahsa Sayyary Namin

First week

In all of the following problems we are working over the field of real numbers. (Note that the equation $x^2+1=0$ has no solution over $\mathbb{R}$, however, it has two solutions $\pm i$ over $\mathbb{C}$)

Problem 1. Determine whether the following systems of equations have any solutions? If no, explain why, and if yes write all the solutions.

$$\begin{cases} 2x+y=0, \\ y+x=1, \\ 2x-y=0. \end{cases}$$

$$\begin{cases} 2x_1-3x_2-7x_3+5x_4+2x_5=-2, \\ x_1-2x_2-4x_3+3x_4+x_5=-2, \\ 2x_1-4x_2+x_5+2x_4=3, \\ x_1-5x_2-7x_3+6x_4+2x_5=-7 \end{cases}$$

Solution 1. The first system has no solution because adding up the first and the last equations result in

$$\begin{aligned} 4x=0 &\Rightarrow x=0 \\ &\Rightarrow y=0 \qquad\qquad (since\ 2x+y=0) \\ &\Rightarrow 1=0 \qquad\qquad (since\ x+y=0) \end{aligned}$$

r_i corresponds to the i-th equation.

solve it, we can form the reduced row echelon form by doing row operations:

$$\begin{array}{cc} & \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & \end{matrix} \\ \begin{matrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{matrix} & \left(\begin{array}{ccccc|c} 2 & -3 & -7 & 5 & 2 & -2 \\ 1 & -2 & -4 & 3 & 1 & -2 \\ 2 & 0 & -4 & 2 & 1 & 3 \\ 1 & -5 & -7 & 6 & 2 & -7 \end{array}\right) \end{array} \xrightarrow{r_1 \leftrightarrow r_2} \begin{array}{cc} \begin{matrix} r_2 \\ r_1 \\ r_3 \\ r_4 \end{matrix} & \left(\begin{array}{ccccc|c} 1 & -2 & -4 & 3 & 1 & -2 \\ 2 & -3 & -7 & 5 & 2 & -2 \\ 2 & 0 & -4 & 2 & 1 & 3 \\ 1 & -5 & -7 & 6 & 2 & -7 \end{array}\right) \end{array}$$

$$\xrightarrow{\substack{r_1 = r_1 - 2r_2 \\ r_3 = r_3 - 2r_2 \\ r_4 = r_4 - r_2}} \begin{array}{cc} \begin{matrix} r_2 \\ r_1 \\ r_3 \\ r_4 \end{matrix} & \left(\begin{array}{ccccc|c} 1 & -2 & -4 & 3 & 1 & -2 \\ 0 & 1 & 1 & -1 & 0 & 2 \\ 0 & 4 & 4 & -4 & -1 & 7 \\ 0 & -3 & -3 & 3 & 1 & -5 \end{array}\right) \end{array}$$

$$\xrightarrow{\substack{r_2 = r_2 + 2r_1 \\ r_3 = r_3 - 4r_1 \\ r_4 = r_4 + 3r_1}} \begin{array}{cc} \begin{matrix} r_2 \\ r_1 \\ r_3 \\ r_4 \end{matrix} & \left(\begin{array}{ccccc|c} 1 & 0 & -2 & 1 & 1 & 2 \\ 0 & 1 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right) \end{array}$$

$$\xrightarrow{r_3 = -r_3} \begin{array}{cc} \begin{matrix} r_2 \\ r_1 \\ r_3 \\ r_4 \end{matrix} & \left(\begin{array}{ccccc|c} 1 & 0 & -2 & 1 & 1 & 2 \\ 0 & 1 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right) \end{array} \xrightarrow{r_4 = r_4 - r_3} \begin{array}{cc} \begin{matrix} r_2 \\ r_1 \\ r_3 \\ r_4 \end{matrix} & \left(\begin{array}{ccccc|c} 1 & 0 & -2 & 1 & 1 & 2 \\ 0 & 1 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

The simplified version of the original system of equations is:

$$\begin{cases} x_1 - 2x_3 + x_4 = 1 \\ x_2 + x_3 - x_4 = 2 \\ x_5 = 1 \end{cases}$$

By setting $x_3 = \alpha$ and $x_4 = \beta$, the solution set is

$$\{(2\alpha - \beta + 1, -\alpha + \beta + 2, \alpha, \beta, 1) \in \mathbb{R}^5 : \ \alpha, \beta \in \mathbb{R})\}$$

Problem 2. Let $A = \begin{pmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, and $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$. For which values of b the system $Ax = b$ has a solution?

Solution 2. We start with simplifying the system as in the previous problem:

$$\begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array}\left(\begin{array}{ccc|c} 3 & -1 & 2 & b_1 \\ 2 & 1 & 1 & b_2 \\ 1 & -1 & 1 & b_3 \end{array}\right) \xrightarrow{r_1 \longleftrightarrow r_3} \begin{array}{c} r_3 \\ r_2 \\ r_1 \end{array}\left(\begin{array}{ccc|c} 1 & -1 & 1 & b_3 \\ 2 & 1 & 1 & b_2 \\ 3 & -1 & 2 & b_1 \end{array}\right)$$

$$\xrightarrow{\substack{r_2 = r_2 - 2r_3 \\ r_1 = r_1 - 3r_3}} \begin{array}{c} r_3 \\ r_2 \\ r_1 \end{array}\left(\begin{array}{ccc|c} 1 & -2 & 1 & b_3 \\ 0 & 5 & -1 & b_2 - 2b_3 \\ 0 & 5 & -1 & b_2 - 3b_3 \end{array}\right) \xrightarrow[r_1 = r_1 - r_2]{} \begin{array}{c} r_3 \\ r_2 \\ r_1 \end{array}\left(\begin{array}{ccc|c} 1 & -2 & 1 & b_3 \\ 0 & 5 & -1 & b_2 - 2b_3 \\ 0 & 0 & 0 & b_1 + b_2 - 5b_3 \end{array}\right)$$

for $\{b \in \mathbb{R}^3 : b_1 + b_2 = 5b_3\}$, the system $Ax = b$ has (at least) one solution.

Problem 3. Find the determinant of the following matrices:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 4 & 8 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ -1 & 0 & 3 & \dots & n-1 & n \\ -1 & -2 & 0 & \dots & n-1 & n \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -1 & -2 & -3 & \dots & -(n-1) & 0 \end{pmatrix}$$

Remark 1. Based on your questions in the class: Determinant is ONLY defined for square matrices.

Solution 3. The first equation shows how to find the determinant of a 2×2 matrix. The second one shows how to reduce the problem of finding a 3×3 determinant to the 2×2 case. The third equation reduces finding a 4×4 determinant to 3×3 determinants. Therefore you can see that following this pattern you can see how to find a $n \times n$ determinant by finding some number of 2×2 determinants.

3 is in the row $\boxed{1}$ and column $\boxed{3}$. $(-1)^{\boxed{1+3}} = +1$ therefore the sign is possitive

$$\det A = +1 \times 4 - 2 \times 3 = -2$$

$$\det B = +1 \times \det \begin{pmatrix} 2 & 1 \\ 4 & 8 \end{pmatrix} - 2 \times \det \begin{pmatrix} 3 & 1 \\ 2 & 8 \end{pmatrix} + 3 \times \det \begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix} = \dots$$

here we expand with respect to the last row because of the cancellations..

$$\det C = -1 \times \begin{pmatrix} 2 & 3 & 4 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix} + 0 \times \begin{pmatrix} 1 & 3 & 4 \\ 4 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} - 1 \times \begin{pmatrix} 1 & 2 & 4 \\ 4 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$+ 0 \times \begin{pmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 0 & 1 & 0 \end{pmatrix} = \dots$$

0

0

Note that:

- Determinant of a triangular matrix is the product of the elements on the diagonal,
- Row and column operations do not change the determinant.

By adding the first row of the matrix D to all the other rows you get the triangular matrix:

$$\begin{pmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ 0 & 2 & \star & \dots & \star & \star \\ 0 & 0 & 3 & \star & \star & \star \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & n \end{pmatrix}$$

the star elements are not important.

Therefore $\det D = 1 \times 2 \times 3 \times \dots n = n!$

Problem 4. Let A and B be two $n \times n$ matrices. If $\det A = 5$ and $detB = 7$, compute $\det(A^2B^3A)$.

Solution 4.

$$\begin{aligned}\det(A^4B^3A) &= \det(A^5B^3)\\ &= \det(A^5) \times \det(B^3)\\ &= (\det A)^5 \times (\det B)^3\\ &= 5^5 \times 7^3.\end{aligned}$$

Note that:

- If A and B are two matrices such that AB and BA are both defined, then $\det AB = \det BA$,
- If A and B are two matrices of size $n \times n$ then $\det AB = \det A \times \det B$

Problem 5. Let $A = \begin{pmatrix} a & 0 & 0 \\ d & b & 0 \\ f & e & c \end{pmatrix}$ such that a, b, and c are nonzero, compute A^{-1}.

Remark 2. Based on your questions in the class: Inverse is ONLY defined for square matrices with nonzero determinant.

Solution 5. In this problem we show that the inverse of a lower triangular 3×3 matrix (if exists) is lower triangular and the entries on the diagonal of A^{-1} are the inverse of the corresponding entries on the diagonal of A. Moreover you can see a general method to find the inverse of a matrix.

Since none of a, b, or c are zero, $\det A = abc \neq 0$ and the inverse A^{-1} exists. We attach an identity matrix of the proper size to A and make the reduced row echelon form of A that is going to be the identity matrix. The identity matrix that we attached at the beginning will turn to A^{-1} by this process.

$$\begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array}\left(\begin{array}{ccc|ccc} a & 0 & 0 & 1 & 0 & 0 \\ d & b & 0 & 0 & 1 & 0 \\ f & e & c & 0 & 0 & 1 \end{array}\right) \xrightarrow{r_1 = r_1/a} \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{a} & 0 & 0 \\ d & b & 0 & 0 & 1 & 0 \\ f & e & c & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{\substack{r_2 = r_2 - d.r_1 \\ r_3 = r_3 - f.r_1}} \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{a} & 0 & 0 \\ 0 & b & 0 & \frac{-d}{a} & 1 & 0 \\ 0 & e & c & \frac{-f}{a} & 0 & 1 \end{array}\right)$$

$$\xrightarrow{r_2 = r_2/b} \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{a} & 0 & 0 \\ 0 & 1 & 0 & \frac{-d}{ab} & \frac{1}{b} & 0 \\ 0 & e & c & \frac{-f}{a} & 0 & 1 \end{array}\right)$$

$$\xrightarrow{r_3 = r_3 - e.r_2} \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{a} & 0 & 0 \\ 0 & 1 & 0 & \frac{-d}{ab} & \frac{1}{b} & 0 \\ 0 & 0 & c & \frac{-f}{a} + \frac{de}{ab} & \frac{-e}{b} & 1 \end{array}\right) \xrightarrow{r_3 = r_3/c} \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{a} & 0 & 0 \\ 0 & 1 & 0 & \frac{-d}{ab} & \frac{1}{b} & 0 \\ 0 & 0 & 1 & \frac{-f}{ac} + \frac{de}{abc} & \frac{-e}{bc} & \frac{1}{c} \end{array}\right)$$

$$\text{Therefore } A^{-1} = \begin{pmatrix} \frac{1}{a} & 0 & 0 \\ \frac{-d}{ab} & \frac{1}{b} & 0 \\ \frac{-f}{ac} + \frac{de}{abc} & \frac{-e}{bc} & \frac{1}{c} \end{pmatrix}.$$

Problem 6. Let A be the matrix from the previous problem. Does the system $Ax = b$ have a solution? Is the solution unique?

Solution 6. Yes, it has a solution and it is unique. Let $v = A^{-1}b$ then $A(A^{-1}b) = b$. Thus v is a solution. Now let w be a solution. We have $Aw = b$. Multiplying both sides by A^{-1} gives $w = A^{-1}b = v$. Therefore the solution is unique.

Remark 3. If the matrix A is invertible then the system $Ax = b$ has a unique solution. If the matrix A is not invertible the system can either have no solution or infinity many solutions. We have discussed both cases in the class.

Problem 7. For which values of α and β, the vectors $v_1 = (\alpha, 1)$ and $v_2 = (0, \beta)$ form a basis for $\mathbb{R}^2$. In other words, v_1 and v_2 are linearly independent.

Solution 7. Two vectors v_1 and v_2 forming a basis for $\mathbb{R}^2$ is equivalent to $\det \begin{pmatrix} \alpha & 1 \\ 0 & \beta \end{pmatrix} \neq 0$. Therefore, we need $\alpha \neq 0$ and $\beta \neq 0$.

Problem 8. Use the alternating symbol (Levi-Civita symbol) ϵ_{ijk} and the Kronecker delta δ_{ij} to show

$$u.(v \times w) = v.(w \times u) = w.(u \times v).$$

Solution 8. To be discussed in our next meeting.