Series 8

1. Let $\mathcal{T}$ be the system of open subsets of $\mathbb{R}^N$ und $s > 0$. Let $\tau_s : \mathcal{T} \to [0, \infty]$ be the function

$$\tau_s(U) \;=\; \big(\operatorname{diam}(U)\big)^s\,.$$

Let $\mathcal{T}_\varepsilon = \{\, T \in \mathcal{T} \mid \operatorname{diam} T < \varepsilon \,\}$ and consider the outer measure defined by

$$\mu^*_{\varepsilon,s}(A) \;:=\; \inf\Big\{\, \sum_{n=1}^{\infty} \tau_s(U_n) \,\Big|\, (U_n) \subset \mathcal{T}_\varepsilon,\, A \subset \bigcup_{n=1}^{\infty} U_n \,\Big\}\,.$$

a) *Show*: $\mu^*_{\varepsilon',s}(E) \geq \mu^*_{\varepsilon,s}(E)$ for $0 < \varepsilon' < \varepsilon$, $E \subset \mathbb{R}^N$, and deduce that $\mu^*_{(s)}(E) := \lim_{\varepsilon\to 0} \mu^*_{\varepsilon,s}(E)$ is well-defined f.a. $E \subset \mathbb{R}^N$. *(1 pt)*

b) *Show*: $\mu^*_{(s)}$ ist an outer measure on $\mathbb{R}^N$. *(2 pts)*

c) *Show*: For all $T \in \mathcal{T}$, we have

$$\mu^*_{(s)}(T) \;=\; \begin{cases} 0, & \text{if } s > N, \\ \infty, & \text{if } s < N\,. \end{cases}$$

(3 pts)

d) Let $E \subset \mathbb{R}^n$ with $\mu^*_{(s)}(E) < \infty$ and $t > s$. *Show*: $\mu^*_{(t)}(E) = 0$. *(3 optional pts)*

Given $E \subset \mathbb{R}^N$ we call

$$\dim_{HD} E \;=\; \sup\{\, s > 0 \mid \mu^*_{(s)}(E) = \infty \,\}$$

the **Hausdorff-dimension** of E, where $\dim_{HD} E := 0$ if $\mu^*_{(s)}(E) = 0$ f.a. $s > 0$.

Hence, $\dim_{HD} \mathbb{R}^N = N$.

2. Consider the Cantor-set $C = \{\, \sum_{n=1}^{\infty} \frac{a_n}{3^n} \mid a_n \in \{0, 2\} \text{ f.a. } n \in \mathbb{N} \,\}$.

a) Show that C is in the Borel-σ-algebra of $\mathbb{R}$. *Hint: Write C as the intersection of compact subsets $C_1 \supset C_2 \supset \ldots$.* *1 pt*

b) Show explicitly that the 1-dimensional Lebesgue measure $\lambda^1(C)$ of C is 0. (Do not use problem 1 above.) *1 pt*

c) Show that for $s = \log 2/\log 3$ we have for the outer Hausdorff measure $\mu^*_{(s)}$ from Problem 1 that

$$\mu^*_{(s)}(C) \leq 1\,.$$

2 pts

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d) Show that $\mu^*_{(s)}(C) \geq \frac{1}{2}$ and deduce from Problem 1 that $\dim_{HD}(C) = \log 2/\log 3$. *2 optional points*

3. Decide whether the function $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{\sin x}{x}$ for $x \neq 0$ is Lebesgue-integrable or not and prove your answer. *4 pts*

4. **a)** Consider the following sequences of functions $f_n\colon \mathbb{R} \to \mathbb{R}$ and decide and prove whether $\lim_{n\to\infty} \int_{\mathbb{R}} f_n d\lambda^1$ exists and whether $\int_{\mathbb{R}} \lim f_n d\lambda^1 = \lim \int_{\mathbb{R}} f_n d\lambda^1$:

- $\frac{1}{(1+x^2)^n}$
- $f_n(x) = \begin{cases} 0, & x \leq n - \frac{1}{n}, \\ nx + 1 - n^2, & n - \frac{1}{n} \leq x \leq n, \\ 1 + n^2 - nx, & n \leq x \leq n + \frac{1}{n}, \\ 0, & x \geq n + \frac{1}{n}, \end{cases}$

2 points

b) Decide for $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_{count})$ (μ_{count} is the counting measure) and for $([0,1], \mathcal{B}([0,1]), \lambda^1)$, for which $r, s \in [1, \infty)$ we have $L^r(X, \mu) \subset L^s(X, \mu)$. *2 points*

Merry Christmas and Happy New Year !

Hand-In: Practice Session Wednesday Jan 8