HW 13. Solutions.

Problem $\underline{2}$.

- $$\int_0^2 |1-x|\,dx = \int_0^1 (1-x)\,dx + \int_1^2 (x-1)\,dx$$

$$= x - \frac{x^2}{2}\Big]_0^1 + \frac{x^2}{2} - x\Big]_1^2$$

$$= \left(1-\frac{1}{2}\right) - 0 + \left(\frac{4}{2} - 2\right) - \left(\frac{1}{2} - 1\right) = \boxed{1} \quad \checkmark$$

integral by part.

- $$\int_0^{2\pi} \underbrace{x^2}_{f} \underbrace{\cos x}_{g'}\,dx = fg\Big]_0^{2\pi} - \int_0^{2\pi} f'g$$

$$= \cancel{x^2 \sin x\Big]_0^{2\pi}} \to 0 - \int_0^{2\pi} 2x \sin x\,dx = -2\int_0^{2\pi} \underbrace{x}_{u} \underbrace{\sin x}_{v'}\,dx$$

$$= -2\left[uv\Big]_0^{2\pi} - \int_0^{2\pi} u'v\right]$$

$$= -2\left[x(-\cos x)\Big]_0^{2\pi} - \int_0^{2\pi} 1\cdot(-\cos x)\,dx\right]$$

$$= -2\left[-2\pi - 0 + \int_0^{2\pi} \cos x\,dx\right]$$

$$= -2\left(-2\pi + \cancel{\sin x\Big]_0^{2\pi}} \to 0\right) = \boxed{4\pi} \quad \checkmark$$

change of variable

- $\displaystyle\int_0^{\ln 2} \sqrt{e^x - 1}\, dx$

Let $u := e^x - 1 \Rightarrow \dfrac{du}{dx} = e^x = u+1 \Rightarrow dx = \dfrac{du}{u+1}$

Now substitute these in the original integral.

Note that the boundary of your integral also needs to be changed accordingly.

$x = 0 \Rightarrow u = e^0 - 1 = 1-1 = 0$

$x = \ln 2 \Rightarrow u = e^{\ln 2} - 1 = 2-1 = 1$

$$\Rightarrow \int_0^{\ln 2} \sqrt{e^x-1}\, dx = \int_0^1 u^{1/2} \cdot \frac{du}{u+1} = \int_0^1 \frac{u^{1/2}}{u+1} du$$

(Again, let $v = u^{1/2}$, $\dfrac{dv}{du} = \dfrac{1}{2} u^{-1/2} \Rightarrow du = 2u^{1/2} dv = 2v\,dv$

$u = 0 \Rightarrow v = 0$
$u = 1 \Rightarrow v = 1$ $\rightsquigarrow$ here, the boundary doesn't change.)

$$= \int_0^1 \frac{v}{v^2+1} \cdot 2v\,dv = 2\int_0^1 \frac{v^2}{v^2+1} dv$$

(Here we need to use a small trick:

$$\frac{v^2}{v^2+1} = \frac{v^2+1-1}{v^2+1} = 1 - \frac{1}{v^2+1}\Bigg)$$

$$= 2\int_0^1 \left(1 - \frac{1}{v^2+1}\right) dv = 2\left(v\Big]_0^1 - \int_0^1 \frac{1}{v^2+1} dv \right)$$

$$= 2\left(1 - \operatorname{Arctan}(v)\Big]_0^1\right) = 2\left(1 - \frac{\pi}{4}\right) = \boxed{2 - \frac{\pi}{2}}$$ //

change of variable.

$$\bullet \int_2^3 \frac{1}{6^x+1}\,dx$$

Let $u = 6^x + 1$, then $\frac{du}{dx} = \ln 6 \cdot 6^x = \ln 6 \cdot (u-1)$

$$\Rightarrow dx = \frac{1}{\ln 6 \cdot (u-1)}\,du$$

$x = 2 \Rightarrow u = 6^2 + 1$

$x = 3 \Rightarrow u = 6^3 + 1$

$$\int_2^3 \frac{1}{6^x+1}\,dx = \int_{6^2+1}^{6^3+1} \frac{1}{u} \cdot \frac{1}{\ln 6 \cdot (u-1)}\,du$$

$$= \frac{1}{\ln 6} \int_{6^2+1}^{6^3+1} \frac{1}{u(u-1)}\,du$$

(we need a small trick here: $\frac{1}{u(u-1)} = \frac{1}{u-1} - \frac{1}{u}$)

$$= \frac{1}{\ln 6} \cdot \int_{6^2+1}^{6^3+1} \left(\frac{1}{u-1} - \frac{1}{u}\right) du$$

$$= \frac{1}{\ln 6} \Big[\ln(u-1) - \ln u \Big]_{6^2+1}^{6^3+1}$$

$$= \frac{1}{\ln 6} \cdot \left[\left(\ln 6^3 - \ln(6^3+1) \right) - \left(\ln 6^2 - \ln(6^2+1) \right) \right]$$

$$= \frac{1}{\ln 6} \cdot \left(\underbrace{3\ln 6 - \ln(6^3+1) - 2\ln 6}_{\ln 6} + \ln(6^2+1) \right)$$

$$= 1 + \frac{1}{\ln 6} \left(\ln(6^2+1) - \ln(6^3+1) \right)$$

Problem 4.

- $x^{10} \ln x$

x^{10} is defined everywhere and $\ln x$ is defined on $(0, +\infty)$. Therefore, the domain of $x^{10}\ln x$ is $(0, +\infty)$ and this is where we find the antiderivative.

$$\int \underbrace{x^{10}}_{f'} \underbrace{\ln x}_{g} \, dx = fg - \int f g' \quad \text{(integral by part)}$$

$$= \frac{x^{11}}{11} \cdot \ln x - \int \frac{x^{11}}{11} \cdot \frac{1}{x} dx = \frac{1}{11} x^{11} \ln x - \frac{1}{11} \int x^{10} dx$$

$$= \frac{1}{11} \left(x^{11} \ln x - \frac{x^{11}}{11} \right) = \underbrace{\frac{x^{11}}{11} \left(\ln x - \frac{1}{11} \right)}_{G :=}$$

If you take G' on $(0, +\infty)$ you will get $x^{10}\ln x$.

- $\sin^5 x \cos x$

$\sin x$ and $\cos x$ are defined on all $\mathbb{R}$ therefore the domain of $\sin^5 x \cos x$ is $\mathbb{R}$.

$$\int \sin^5 x \cos x \, dx = \int u^5 \, du = \frac{u^6}{6} = \frac{1}{6} \sin^6 x$$

change of variable $u = \sin x$, $\frac{du}{dx} = \cos x \Rightarrow du = \cos x \, dx$

Observe that on $\mathbb{R}$ $\left(\frac{1}{6} \sin^6 x\right)' = \sin^5 x \cos x$.

- $\sin^4 x$

$\sin x$ is defined everywhere and thus the domain of $\sin^4 x$ is $\mathbb{R}$.

integral by part.

$$\int \sin^4 x = \int \overbrace{\sin^3 x}^{f} \, \overbrace{\sin x}^{g'} \, dx = fg - \int f'g$$

$$= \sin^3 x \cdot (-\cos x) - \int 3 \sin^2 x \cos x \cdot (-\cos x) dx$$

$$= -\sin^3 x \cos x + 3 \int \sin^2 x \overbrace{\cos^2 x}^{(1 - \sin^2 x)} dx$$

$$= -\sin^3 x \cos x + 3 \int \sin^2 x (1 - \sin^2 x) \, dx$$

$$= -\sin^3 x \cos x + 3 \int (\sin^2 x - \sin^4 x) \, dx$$

$$= -\sin^3 x \cos x + 3\int \sin^2 x\,dx - 3\underbrace{\int \sin^4 x\,dx}$$

Now observe that the same integral appeared again. You can rewrite it again:

$$\int \sin^4 x\,dx = -\sin^3 x\cos x + 3\int \sin^2 x\,dx - 3\int \sin^4 x\,dx$$

take it to the other side

$$\Rightarrow 4\int \sin^4 x\,dx = -\sin^3 x\cos x + 3\int \sin^2 x\,dx$$

and $\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx = \int \frac{1}{2} - \int \frac{\cos 2x}{2}$

$$= \frac{x}{2} - \frac{1}{2}\int \cos 2x = \frac{x}{2} - \frac{1}{2}\left(\frac{1}{2}\sin 2x\right)$$

$$= \frac{x}{2} - \frac{1}{4}\sin 2x$$

$$\Rightarrow 4\int \sin^4 x\,dx = -\sin^3 x\cos x + 3\left(\frac{x}{2} - \frac{1}{4}\sin 2x\right)$$

$$\Rightarrow \int \sin^4 x\,dx = \underbrace{\frac{-1}{4}\sin^3 x\cos x + \frac{3}{4}\left(\frac{x}{2} - \frac{1}{4}\sin 2x\right)}_{G}.$$

Use a computer to check that the derivative of G is $\sin^4 x$ everywhere on $\mathbb{R}$.

Problem 6.

Review of what you have in your lecture notes:

Jensen inequality: Let $\sum_{i=1}^{n} \theta_i = 1$ and $\theta_i \in [0,1]$.

Then for a convex function $\varphi: \mathbb{R} \to \mathbb{R}$ and $x_i \in \mathbb{R}$

one has
$$\varphi\left(\sum_{i=1}^{n} \theta_i x_i \right) \leq \sum_{i=1}^{n} \theta_i \varphi(x_i)$$

And you use this inequality to prove Lemma 3.3,

that is Jensen inequality for integrals:

Lemma 3.3. Let $\varphi: \mathbb{R} \to \mathbb{R}$ be convex and f

be Riemann integrable on $[0,1]$ Then

$$\varphi\left(\int_0^1 f \right) \leq \int_0^1 \varphi(f) .$$

For the more general case, we want to know what is the general form of the above inequality if instead of f being Rieman integrable over $[0,1]$ it was Riemann integrable over $[a,b]$ for arbitrary $a,b \in \mathbb{R}$ $a<b$.

make sure to understand the proof of Lemma 3.3 first. The proof of general case is similar.

General statement: Let $\varphi : \mathbb{R} \to \mathbb{R}$ be convex and f be Riemann integrable over $[a,b]$, $a,b \in \mathbb{R}$, $a<b$.

show that

$$\varphi\left(\frac{\int_a^b f}{b-a}\right) \leq \frac{\int_a^b \varphi(f)}{b-a}.$$

Proof. As you can see, the difference is just a matter of scaling. We write the proof following the one in your Lecture notes:

Every convex function is continuous + Thm B.7 $\Rightarrow$ $\varphi(f)$ is Riemann-Integrable.

For any partition P of $[a,b]$ we have

$$\varphi\left(\frac{L(f,P)}{b-a}\right) = \varphi\left(\frac{\sum_{i=1}^{n-1} \Delta x_i \inf_{[x_i, x_{i+1}]} f}{b-a}\right)$$

$$= \varphi\left(\sum_{i=1}^{n-1} \frac{\Delta x_i}{b-a} \inf_{[x_i, x_{i+1}]} f\right)$$

now call $\frac{\Delta x_i}{b-a} := \theta_i$, Observe that $\sum_{i=1}^{n-1} \theta_i = 1$

therefore we can use Jensen inequality as before:

$$\leq \sum_{i=1}^{n-1} \frac{\Delta x_i}{b-a} \varphi\left(\inf_{[x_i, x_{i+1}]} f\right)$$

$$\leq \frac{1}{b-a} \sum_{i=1}^{n-1} \Delta x_i \sup_{[x_i, x_{i+1}]} \varphi(f) \leq \frac{1}{b-a} U(\varphi(f), P)$$

Thus, for any partition P of $[a,b]$ we have shown:

$$\varphi\left(\frac{L(f,P)}{b-a}\right) \leq \frac{U(\varphi(f), P)}{b-a}.$$

As before, function $\varphi(f)$ is Riemann integrable,

so RHS gets arbitrarily close to $\frac{\int_a^b \varphi(f)}{b-a}$.

f is Riemann integrable and φ is continuous, thus, LHS gets arbitrarily close to $\varphi\left(\frac{\int_a^b f}{b-a}\right)$.

$$\Rightarrow \varphi\left(\frac{\int_a^b f}{b-a}\right) \le \frac{\int_a^b \varphi(f)}{b-a}.$$

For problem 8 which is a star problem you need slightly more materials than those in your lecture notes. Please discuss them in your Q&A sessions with Dr. Burczak.

HW 13 Solutions

1. $f(x) = x^2$ is C.T.S on $[-1,1] \Rightarrow f \in \mathcal{R}([-1,1])$

$\overset{B.3}{\Longrightarrow} \quad \forall \varepsilon > 0 \quad \exists P_\varepsilon([-1,1]) \quad U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$

one can show that this statement is equivalent

to $\lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} L(f, P_n)$

for ~~a~~ the partition P_n of the interval.

By linearity $\int_{-1}^{1} x^2 dx = \int_{-1}^{0} x^2 dx + \int_{0}^{1} x^2 dx$.

Since x^2 is strictly increasing on $[0,1]$

Taking $P_n = \{0 = x_0 < x_1 < x_2 < \dots < x_n = 1\}$

with $x_{k+1} - x_k = \frac{1}{n}$ (we have equally distanced subintervals)

then on any interval $[x_{k-1}, x_k]$ we have

$$\inf_{x \in [x_{k-1}, x_k]} f(x) = f(x_{k-1})$$

x^2 is increasing on $[0,1]$

Now we can write :

$$L(f,P_n) = \sum_{k=1}^{n} \inf_{x\in[x_{k-1},x_k]} f(x)\,\Delta x_k$$

$$= \sum_{k=1}^{n} f(x_{k-1})\frac{1}{n} = \sum_{k=1}^{n} (x_{k-1})^2\frac{1}{n}$$

$$= \sum_{k=1}^{n}\left(\frac{k-1}{n}\right)^2\frac{1}{n} = 1/n^3 \sum_{k=1}^{n}(k-1)^2$$

$$= 1/n^3 \sum_{k=1}^{n}(k^2+1-2k)$$

$$= 1/n^3\left(\sum_{k=1}^{n}k^2 + \sum_{k=1}^{n}1 - 2\sum_{k=1}^{n}k\right)$$

$$= 1/n^3\left(\frac{n(n+1)(2n+1)}{6} + n - 2\left(\frac{n(n+1)}{2}\right)\right)$$

$$= 1/n^3\left(n^3/3 - n^2/2 + n/6\right)$$

$$= 1/3 - 1/2n + 1/6n^2$$

Hence $\lim_{n\to\infty} L(f,P_n) = 1/3$

For interval $[-1,0]$ do the Same, but use the fact that $f=x^2$ is decreasing on $[-1,0]$. And you get again $1/3$.

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3\. a) $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x}} = \lim_{p\to 1} \int_0^p \frac{dx}{\sqrt{\underbrace{1-x}_{u}}}$

$dx = -du$

$-\displaystyle\int \frac{du}{\sqrt{u}} = -\frac{u^{1/2}}{1/2} =$

$-2\sqrt{u}$

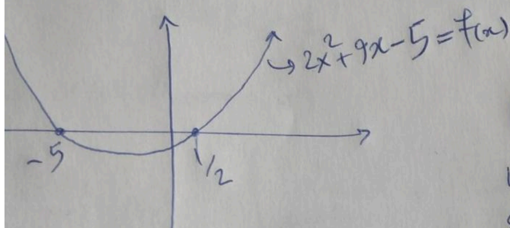
$= \displaystyle\lim_{p\to 1} \left(-2\sqrt{1-x}\,\Big|_0^p\right) = 2$

b) $\displaystyle\int_{-10}^{10} \frac{dx}{2x^2+9x-5} = A$

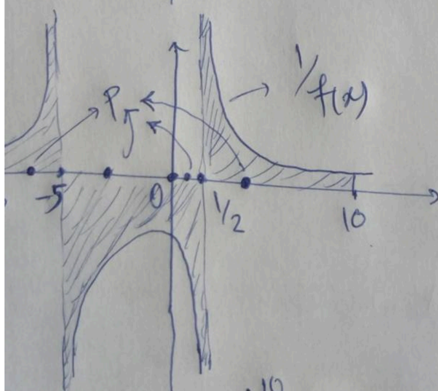
similarly we first find the problematic points :

$2x^2+9x-5=0 \implies x_{1,2} = \dfrac{-9 \pm \sqrt{81-4(2)(-5)}}{4} = \begin{cases} -5 \\ +1/2 \end{cases}$

$\Rightarrow 2x^2+9x-5 = (x+5)(x-1/2)$



Hence, the shaded area must be calculated in many steps.



$A = \displaystyle\lim_{p\to -5^-} \int_{-10}^{p} \frac{dx}{f(x)} +$

$\displaystyle\lim_{p\to -5^+} \int_{p}^{0} \frac{dx}{f(x)} +$

$\displaystyle\lim_{p\to 1/2^-} \int_{0}^{p} \frac{dx}{f(x)} +$

$\displaystyle\lim_{p\to 1/2^+} \int_{p}^{10} \frac{dx}{f(x)}$

(Remember HW 12 Q.1.)

first, let us calculate the indefinite integral

$$\int \frac{dx}{2x^2+9x-5} = \int \frac{dx}{(x+5)(x-1/2)} \;=$$

using partial fraction decomposition we can turn this into a familiar form:

$$\frac{1}{(x+a)(x+b)} = \frac{A}{x+a} + \frac{B}{x+b} = \frac{A(x+b)+B(x+a)}{(x+a)(x+b)}$$

$$\Rightarrow \frac{1}{(x+a)(x+b)} = \frac{x(A+B) + Ab + Ba}{(x+a)(x+b)}$$

$$\Leftrightarrow \begin{cases} A+B=0 \\ Ab+Ba=1 \end{cases} \Rightarrow \begin{cases} A=-B \\ -Bb+Ba=1 \end{cases} \Rightarrow B(a-b)=1$$

$$\Rightarrow B = \frac{1}{a-b}$$

comparison of numerators

$a=5$
$b=-1/2$

$$\Rightarrow \;=\; \int \frac{-1}{\underbrace{(a-b)}_{11/2}(x+5)} + \frac{1}{\underbrace{(a-b)}_{11/2}(x-1/2)} \, dx$$

$$= -\frac{2}{11}\ln|x+5| + \frac{2}{11}\ln|x-1/2|$$

From Here, it is just substitution and taking the limits to see if the integral converges.

you see that for example for the first integral

$$\lim_{p\to -5^-}\int_{-10}^{p} \frac{dx}{f(x)}$$

the expression diverges

$\Rightarrow$ the integral doesn't exist.

3. c) $$\int_0^\infty x^2 e^{-x}\,dx = \lim_{h\to\infty} \int_0^h x^2 e^{-x}\,dx$$

To solve the proper integral we use integration by parts:

$$\int_0^h x^2 e^{-x}\,dx = -x^2 e^{-x}\Big|_0^h + 2\int_0^h x e^{-x}\,dx$$

$du = 2x\,dx \Leftarrow u = x^2$

$v = -e^{-x} \Leftarrow dv = e^{-x}dx$

one more time to killoff the x : $\begin{cases} u = x \\ dv = e^{-x}dx \end{cases}$

$$\Rightarrow \int_0^h x e^{-x}\,dx = -x e^{-x}\Big|_0^h + \int_0^h e^{-x}\,dx$$

$$\left.-e^{-x}\right]_0^h = -e^{-h} + 1$$

$$\Rightarrow \int_0^h x^2 e^{-x}\,dx = -h^2 e^{-h} + 2\left(-h e^{-h} + \int_0^h e^{-x}\,dx\right)$$

$$= -h^2 e^{-h} + 2 - 2e^{-h} - 2he^{-h}$$

$$\Rightarrow \lim_{h\to\infty}\left(2 - \underbrace{e^{-h}(2h + h^2 + 2)}\right) = 2$$

$$\lim_{h\to\infty} \frac{h^2 + 2h + 2}{e^h} = 0$$

Problem 3 : Does $\int_0 x^p e^{-x} dx$ converge for $p<0$?

to use the ~~the~~ Direct Comparison test, we need to come up with functions that converge or diverge in order to be useful. The functions x^p play an important role in Direct compariso. So, we comput $\int_0^1 x^p dx$ and $\int_1^\infty x^p dx$ for $\underline{p>-1,\ p<-1,\ p=-1}$

$p=-1$:

$$\int_0^1 \frac{1}{x} dx = \lim_{b \to 0^+} \int_b^1 \frac{1}{x} dx = \lim_{b \to 0^+} \left(\ln|x| \Big|_b^1 \right)$$

$$= \underbrace{\ln|1|}_{0} - \underbrace{\lim_{b \to 0^+} \ln|b|}_{-\infty} = +\infty$$

$p \neq -1$:

$$\int_0^1 x^p dx = \lim_{b \to 0^+} \int_b^1 x^p dx = \lim_{b \to 0^+} \left(\frac{x^{p+1}}{p+1} \Big|_b^1 \right)$$

$$= \lim_{b \to 0^+} \left(\frac{1}{p+1} - \frac{b^{p+1}}{p+1} \right)$$

Now if $p<-1$ then $b^{\overset{<0}{p+1}} = \dfrac{1}{b^{-p-1}}$ $\longrightarrow >0$

and $\lim_{b \to 0^+} \dfrac{1}{b^{-p-1}}$ Does not exist.

And if $p>-1$, $b^{p+1} \xrightarrow{b \to 0^+} 0$

therefore,

$$\int_0^1 x^p\,dx = \begin{cases} \dfrac{1}{p+1} & \text{if } p>-1 \\ \text{Divergent} & \text{if } p<-1 \end{cases}$$

Now, we look at $\int_1^\infty x^p\,dx$:

$$\int_1^\infty x^p\,dx = \lim_{b\to\infty} \int_1^b x^p\,dx = \lim_{b\to\infty} \left(\frac{x^{p+1}}{p+1} \Big|_1^b \right)$$

$$= \lim_{b\to\infty} \left(\frac{b^{p+1}}{p+1} - \frac{1}{p+1} \right) \quad (*)$$

Hence, if $p>-1 \Rightarrow b^{p+1} \xrightarrow{b\to\infty} \infty$

and $\lim_{b\to\infty} \frac{b^{p+1}}{p+1}$ "DNE"

for $p<-1$ we have $\frac{b^{p+1}}{p+1} = \frac{1}{(p+1)\, b^{-(p+1)}}$

and $\lim_{b\to\infty} \frac{1}{b^{-p-1}} = 0$

$$\Rightarrow \int_1^\infty x^p\,dx = \begin{cases} \text{Divergent} & p>-1 \\ \dfrac{1}{1-p} & p<-1 \end{cases}$$

So now we can use the results.

Case 1: $p = -1$

$$\int_0^\infty \frac{1}{x} e^{-x} dx = \int_0^1 \frac{1}{x} e^{-x} dx + \int_1^\infty \frac{1}{x} e^{-x} dx$$

$\forall x \in [0,1] \quad e^x \leq e^1 \Rightarrow 0 \leq \frac{1}{xe} \leq \frac{1}{xe^x}$

and $\int_0^1 \frac{1}{xe} dx = \frac{1}{e} \underbrace{\int_0^1 \frac{1}{x} dx}_{\text{Divergent}}$

$\Rightarrow \int_0^1 \frac{1}{x} e^{-x} dx$ Diverges.

Case 2: $p < -1$

$\forall x \in [0,1] \quad x^p \geq x^{-1} \Rightarrow \forall x \in [0,1] \quad x^p \cdot e^{-x} \geq x^{-1} \cdot e^{-x} \geq 0$

$\Rightarrow$ Since $\int_0^1 x^{-1} e^{-x} dx$ Diverges,

$\int_0^1 x^p e^{-x} dx$ Diverges as well

Case 3: $p > -1$; $\int_0^\infty x^p e^{-x} dx = \int_0^1 x^p e^{-x} dx + \int_1^\infty x^p e^{-x} dx$

i) $\forall x \in [0,1]$ $\quad 0 \leqslant x^p \cdot e^{-x} \leqslant x^p \cdot 1$

and $\displaystyle\int_0^1 x^p dx = \frac{1}{p+1}$ for $p > -1$

ii) Now we look at $\displaystyle\int_1^\infty x^p e^{-x} dx$!

$$\int_1^\infty x^p e^{-x} dx = \underbrace{\int_1^b x^p e^{-x} dx}_{\text{finite}} + \int_b^\infty x^p e^{-x} dx$$

$\left(\frac{x^p}{e^x} < x^p \right.$ when $x \geq 1 \Rightarrow \int_1^b x^p e^{-x} dx \leq \int_1^b x^p dx$

now check $(*)$ to see the finiteness. $\left.\right)$

Since exponential function grows faster we can write : $\exists b :$ $\quad x^p \leqslant e^{x/2}$ $\quad \forall x \geqslant b$

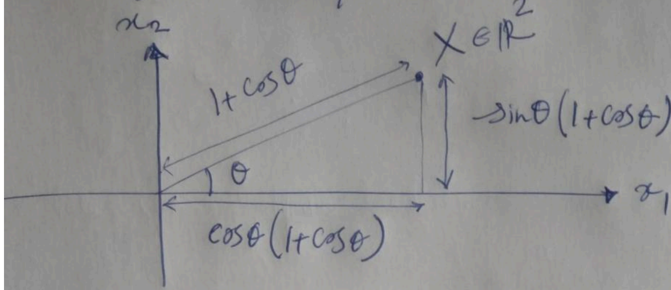
$$\Rightarrow 0 \leqslant x^p \cdot e^{-x} \leqslant e^{x/2} \cdot e^{-x} \qquad \forall x \geqslant b$$

$$\Rightarrow \int_b^\infty e^{x/2} \cdot e^{-x} dx = \int_b^\infty e^{-x/2} dx$$

Now, the final integral $\displaystyle\int_b^\infty e^{-x/2} dx$ is convergent (check!) $\Rightarrow$ $\displaystyle\int_1^\infty x^p e^{-x} dx$ is convergent

$\Rightarrow \displaystyle\int_0^\infty x^p e^{-x} dx$ is convergent !

5. By the description given for the curve we have



Therefore we have $\gamma : \theta \longrightarrow \big(\overbrace{x_1(\theta)}^{\cos\theta(1+\cos\theta)}, \underbrace{x_2(\theta)}_{\sin\theta(1+\cos\theta)}\big)$

$$\ell(\gamma) = \int_0^{2\pi} \| \gamma'(\theta) \| \, d\theta$$

$$\gamma'(\theta) = \left(-\sin\theta - 2\sin\theta\cos\theta \, , \; \cos\theta + \cos^2\theta - \sin^2\theta \right)$$

using Trig identities we have:

$$2\sin\theta\cos\theta = -\sin(2\theta) \; ; \quad \cos^2\theta - \sin^2\theta \overset{(1)}{=} \cos(2\theta)$$

$$\Longrightarrow \quad \| \gamma'(\theta) \| = \sqrt{\left(-(\sin\theta + \sin 2\theta)\right)^2 + \left(\cos\theta + \cos 2\theta\right)^2}$$

$$= \sqrt{\sin^2\theta + \sin^2 2\theta + 2\sin\theta\sin 2\theta + \cos^2\theta + \cos^2 2\theta + 2\cos\theta\cos 2\theta}$$

$$= \sqrt{1+1+2\left(\sin\theta\sin 2\theta + \cos\theta\cos 2\theta\right)}$$

$$= \sqrt{2+2(\cos\theta)} = \sqrt{2(1+\cos\theta)} \overset{*}{=} \sqrt{2\left(2\cos^2 \theta/2\right)}$$

$$= 2\left|\cos \theta/2\right|$$

use $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

for $(*)$ use (1)

$\cos^2\theta - \sin^2\theta =$

$\cos^2\theta - (1 - \cos^2\theta)$

$= 2\cos^2\theta - 1$

$\Rightarrow \cos 2\theta + 1$

$= 2\cos^2\theta$

Therefore we get the Arc length:

$$\ell(\gamma) = 2\int_0^{2\pi} |\cos \theta/2|\, d\theta$$

$$\int_0^{2\pi} |\cos \theta/2|\, d\theta = \int_0^{\pi} |\cos\theta/2|\, d\theta + \int_{\pi}^{2\pi} |\cos\theta/2|\, d\theta$$

$$= \int_0^{\pi} \cos\theta/2\, d\theta - \int_{\pi}^{2\pi} \cos\theta/2\, d\theta$$

$$= \underbrace{\Big[2 \sin\theta/2\Big]_0^{\pi}}_{2} - \underbrace{\Big[2\sin\theta/2\Big]_{\pi}^{2\pi}}_{-2} = 4$$

7. $f:\mathbb{R} \longrightarrow \mathbb{R}$ $\qquad f(a)-f(b)=0$

$\ast$ f' is continuous on $[a,b] \Longrightarrow f \in R([a,b])$

Let's solve the first integral using integration by parts:

$$1 = \int_a^b \underbrace{f^2(x)}_{u} \cdot \underbrace{[redacted]}_{v'} \, dx = \left[x f^2(x)\right]_a^b - \int 2x\, f(x) f'(x)\, dx$$

$$uv\Big]_a^b - \int_a^b u'v$$

$$= \left(b \cancel{f^2(b)} - a \cancel{f^2(a)}\right) - 2\int_a^b x f(x) f'(x)\, dx$$

$$\qquad\quad 0 \qquad\qquad 0$$

$$\Longrightarrow \int_a^b x f(x) f'(x)\, dx = -1/2 \qquad \blacksquare$$

Note: the Hölder inequality for integral can help here since: $(f'(x))^2 = |f'(x)|^2$

and $\quad x^2 f^2(x) = |x f(x)|^2$

$p = 1/2$
$q = 1/2$
$\Rightarrow$

$$\left(\int_a^b |f'(x)|^2 dx\right)^{1/2} \left(\int_a^b |x f(x)|^2 dx\right)^{1/2} \geq$$

$$\left|\int_a^b x f(x) f'(x)\, dx\right| = \frac{1}{2}$$