Hints, Notes and exercises related to HW 14.

Original definition of differential without using the partial derivatives. This we normally use to check the differentiability of a function at a point.

We say that a function $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}^d$ is differentiable at a point x_0 if there exists a linear map $L: \mathbb{R}^d \to \mathbb{R}$

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - L(x - x_0)}{|x - x_0|} = 0$$

Let us use this definition to solve an exercise:

Exercise 1: Let $f: \mathbb{R}^3 \to \mathbb{R}$ be the map:

$$f(x_1, x_2, x_3) = (x_1^2 + x_2^2 + x_3^2)^{\alpha/2}$$

where $\alpha \in [0, \infty)$. For which values of α, f is differentiable at the point $O = (0,0,0) \in \mathbb{R}^3$?

Solution:

we want to have

$$\lim_{(x_1, x_2, x_3) \to (0,0,0)} \frac{f(x_1, x_2, x_3) - f(0,0,0) - L\big((x_1, x_2, x_3) - (0,0,0)\big)}{\big|(x_1, x_2, x_3) - (0,0,0)\big|} = 0$$

i.e. $$\lim_{(x_1,x_2,x_3)\to(0,0,0)} \frac{(x_1^2+x_2^2+x_3^2)^{\alpha/2} - 0 - L(x_1,x_2,x_3)}{(x_1^2+x_2^2+x_3^2)^{1/2}} = 0$$

By looking at the denominator we can make a guess that maybe we need to break the problem in smaller piecess : ($\alpha<1$, $\alpha=1$, $\alpha>1$)

let $X=(x_1,x_2,x_3)$ and $0=(0,0,0)$

let $\alpha>1$: $$\lim_{X\to 0} \frac{(x_1^2+x_2^2+x_3^2)^{\alpha/2} - L(x_1,x_2,x_3)}{(x_1^2+x_2^2+x_3^2)^{1/2}} \quad *$$

If you take $L(X)=0$ then

$$* = \lim_{X\to 0} (x_1^2+x_2^2+x_3^2)^{\frac{\alpha-1}{2}} = 0$$

$\frac{\alpha-1}{2} > 0$

Therefore, for $\alpha>1$ we could find a linear map $L(x)$ such that the limit $*$ is zero. This means f is differentiable at 0.

Small remark: this is a coincidence that $L(X)=0$ and we are looking at the differentiability at 0. These two zeros are not related. :)

Next, Let $\alpha = 1$:

$$* = \lim_{x \to 0} \frac{(x_1^2 + x_2^2 + x_3^2)^{1/2} - L(x_1, x_2, x_3)}{(x_1^2 + x_2^2 + x_3^2)^{1/2}}$$

Note that $L(x_1, x_2, x_3) = a x_1 + b x_2 + c x_3$ for $a, b, c \in \mathbb{R}$

If the limit above exists, it should exist no matter from which direction we go to zero. Set the direction

$x_2 = x_3 = 0$ i.e. $x = (x_1, 0, 0)$

now the limit above is

$$(**) = \lim_{x_1 \to 0} \frac{(x_1^2)^{1/2} - L(x_1)}{(x_1^2)^{1/2}} = \lim_{x_1 \to 0} \frac{|x_1| - a x_1}{|x_1|}$$

$$\nearrow \lim_{x_1 \to 0^+} \frac{x_1 - a x_1}{x_1} = \lim_{x_1 \to 0^+} \frac{x_1(1-a)}{x_1} = \lim_{x_1 \to 0^+} (1-a) = 1-a$$

$$\searrow \lim_{x_1 \to 0^-} \frac{(-x_1) - a x_1}{-x_1} = \lim \frac{-x_1(1+a)}{-x_1} = 1+a$$

$\Rightarrow 1-a = 1+a \Rightarrow a = 0$,

Similarly, you can show that $b=0$ and $c=0$.

$\Rightarrow L(x) = 0$, but then $$* = \lim_{x \to 0} \frac{(x_1^2 + x_2^2 + x_3^2)^{1/2}}{(x_1^2 + x_2^2 + x_3^2)^{1/2}} = 1$$

and is not zero! Thus f is not differentiable at zero for $\alpha = 1$.

Now let $\alpha < 1$. We want to show that there is no

$L(x) = a x_1 + b x_2 + c x_3$ $\quad a, b, c \in \mathbb{R}$ such that

$$\lim_{x \to 0} \frac{(x_1^2 + x_2^2 + x_3^2)^{\alpha/2} - L(x)}{(x_1^2 + x_2^2 + x_3^2)^{1/2}} = 0 . \quad \rightarrow (**)$$

Again pick a direction as before $(x_1, 0, 0) \to 0$

$$(**) = \lim_{x_1 \to 0} \frac{(x_1^2)^{\alpha/2} - a_1 x_1}{(x_1^2)^{1/2}} = \lim_{x_1 \to 0} \frac{|x_1|^{\alpha} - a_1 x_1}{|x_1|}$$

(Since $\alpha < 1$)
$$= \lim_{x_1 \to 0} \frac{|x_1|^{\alpha}}{|x_1|} = \lim_{x_1 \to 0} |x_1|^{\overset{<0}{\alpha - 1}} \to \infty$$

Therefore, f is not differentiable at 0 if $\alpha < 1$.

Once you know that a function is differentiable at a point x_0 you can use its partial derivatives to represent the linear map L with its matrix representation wrt the canonical basis, this is :

$f : U \to \mathbb{R}^n \qquad U \subseteq \mathbb{R}^d$

$(x_1, x_2, \dots, x_d) \longmapsto (f_1(x_1, \dots, x_d), f_2(x_1, \dots, x_d), \dots, f_n(x_1, \dots, x_d))$

Where $f_i : U \to \mathbb{R} \quad i \in \{1, \dots, n\}$

$$\nabla f(x_0) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x_0) & \frac{\partial f_1}{\partial x_2}(x_0) & \cdots & \frac{\partial f_1}{\partial x_d}(x_0) \\ \frac{\partial f_2}{\partial x_1}(x_0) & \frac{\partial f_2}{\partial x_2}(x_0) & \cdots & \frac{\partial f_2}{\partial x_d}(x_0) \\ \vdots & & & \\ \frac{\partial f_n}{\partial x_1}(x_0) & - & - & \frac{\partial f_n}{\partial x_d}(x_0) \end{bmatrix}$$

If you mean ∇f at any point of the domain then you simply write ∇f without evaluating the partial derivatives at a specific point x_0.

$\det(\nabla f)$ is called the jacobian determinant of f.

This is all you need to know to solve Problem 1.

Note that for $\det(\nabla f)$ to be defined, you need ∇f to be a square matrix, which means $d = n$ in the definition of f.

Exercise 2: let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be defined as

$$f(x,y) = (\overbrace{\sin x}^{f_1}, \overbrace{\cos(2xy+y)}^{f_2})$$ what is the differential and the Jacobian determinant of f?

Solution:

$$\nabla f = \begin{bmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} \cos x & 0 \\ -\sin(2xy+y) \times 2y & -\sin(2xy+y) \times (2x+1) \end{bmatrix}$$

$$\det \nabla f = -(2x+1)\cos x \sin(2xy+y)$$

Exercise 3. Let f be $f: \mathbb{R}^3 \to \mathbb{R}^3$

$$(x_1, x_2, x_3) \longmapsto \left(\operatorname{Ln}(x_1 + x_2^2) + x_3 e^{x_1}, \; x_2 + x_3, \; x_3\right)$$

Compute the differential and the Jacobian determinant at the point $p = (0,-1,1)$.

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$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$(x_1, x_2, x_3) \longmapsto \Big(\underbrace{\ln(x_1 + x_2^2) + x_3 e^{x_1}}_{f_1} , \underbrace{x_2 + x_3}_{f_2} , \underbrace{x_3}_{f_3} \Big)$$

$$\nabla f = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{x_1 + x_2^2} + x_3 e^{x_1} & \frac{2x_2}{x_1 + x_2^2} & e^{x_1} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\nabla f(0, -1, 1) = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

upper triangular $\Rightarrow$

$$\det \nabla f(0, -1, 1) = (2)(1)(1) = \boxed{2}$$

for problem 2 you need to use the definition of ∇f as above and Lemma 4.5 in your Lecture notes:

Exercise 4: Let $u : \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function and define $v : \mathbb{R}^2 \to \mathbb{R}$ as

$$v(\alpha, \beta) = u(\sin\alpha, \cos\beta, \alpha^2 + 2\alpha\beta)$$

what is ∇v ?

solution: Define $h : \mathbb{R}^2 \to \mathbb{R}^3$

$$(\alpha, \beta) \mapsto (\underbrace{\sin\alpha}_{h_1}, \underbrace{\cos\beta}_{h_2}, \underbrace{\alpha^2 + 2\alpha\beta}_{h_3})$$

now observe that $v = u \circ h$

Now from Lemma 4.5.

$$\nabla v = \nabla u(h) \times \nabla h \quad (\text{remember} : (f \circ g)' = f'(g) \times g')$$

we do not know what u is but lets assume:

$u : \mathbb{R}^3 \to \mathbb{R}$

$(x, y, z) \mapsto u(x, y, z)$

Then,

$$\Rightarrow \underbrace{\nabla v(\alpha,\beta)}_{\text{diff of } v \text{ at the point } (\alpha,\beta)} = \begin{bmatrix} \frac{\partial u}{\partial x}(h(\alpha,\beta)) & \frac{\partial u}{\partial y}(h(\alpha,\beta)) & \frac{\partial u}{\partial z}(h(\alpha,\beta)) \end{bmatrix}_{1\times 3}$$

$$\times \begin{bmatrix} \frac{\partial h_1}{\partial \alpha} & \frac{\partial h_1}{\partial \beta} \\ \frac{\partial h_2}{\partial \alpha} & \frac{\partial h_2}{\partial \beta} \\ \frac{\partial h_3}{\partial \alpha} & \frac{\partial h_3}{\partial \beta} \end{bmatrix}_{3\times 2}$$

$$= \begin{bmatrix} \frac{\partial u}{\partial x}(\sin\alpha, \cos\beta, \alpha^2+2\alpha\beta) & \frac{\partial u}{\partial y}(\sin\alpha, \cos\beta, \alpha^2+2\alpha\beta) & \frac{\partial u}{\partial z}(\sin\alpha, \cos\beta, \alpha^2+2\alpha\beta) \end{bmatrix}_{1\times 3}$$

$$\times \begin{bmatrix} \cos\alpha & 0 \\ 0 & -\sin\beta \\ 2\alpha+2\beta & 2\alpha \end{bmatrix}_{3\times 2} \qquad (*)$$

You see that ∇f is this matrix multiplication which is a 1×2 matrix. as we want.

To fully understand what is going on let us now set an explicit formula for u (this you do not need to do for your exercises :))

Let $u : \mathbb{R}^3 \to \mathbb{R}$ be $u(x,y,z) = x^2 + 2y^2 + 3z^2$

now evaluate $(*)$:

$$= \begin{bmatrix} 2\sin\alpha & 4\cos\beta & 6(\alpha^2+2\alpha\beta) \end{bmatrix} \times \begin{bmatrix} \cos\alpha & 0 \\ 0 & -\sin\alpha \\ 2\alpha+2\beta & 2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 2\sin\alpha\cos\alpha + 12(\alpha+\beta)(\alpha^2+2\alpha\beta) & -4\sin\alpha\cos\beta + 12\alpha(\alpha^2+2\alpha\beta) \end{bmatrix}$$

Let us now directly compute $u \circ h = v$ and then ∇v just for fun:

$$v(\alpha,\beta) = u(\sin\alpha, \cos\beta, \alpha^2+2\alpha\beta)$$
$$= \sin^2\alpha + 2\cos^2\beta + 3(\alpha^2+2\alpha\beta)^2$$

$$\nabla v = \begin{bmatrix} \frac{\partial v}{\partial \alpha} & \frac{\partial v}{\partial \beta} \end{bmatrix} =$$

!:)

A link to the theorem that you need to use for Problem 3 is on my homepage.

A couple of remarks from your lecture notes:

① If f is differentiable at a point x_0 then its partial derivaties exist. (In particular you can define the differential using the partial derivaties)

* The converse is not necessarily true, the existence of partial derivatieves does not gaurantee the differentiability. (example in lecture notes)

② If the partial derivatives exist and they are all continuous then f is differentiable.

* The converse is not necessarily true. i.e. It could be that f is differentiable at a point x_0 but its partial derivatives are not continuous.

Exercise 5: Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$

such that $f(x,y) = \begin{cases} 0 & x=y=0 \\ (x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right) & \text{otherwise} \end{cases}$

① Show that f is differentiable at 0.

② Compute the partial derivatives for all $(x_0, y_0) \in \mathbb{R}^2$.

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$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - L(x,y)}{|(x,y)-(0,0)|} = \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)\sin\frac{1}{x^2+y^2} - L(x,y)}{(x^2+y^2)^{1/2}}$$

Let $L(x,y) = 0$ then

$$= \lim_{(x,y)\to(0,0)} (x^2+y^2)^{1/2} \sin\frac{1}{x^2+y^2} = (*)$$

$$-1 \leq \sin\frac{1}{x^2+y^2} \leq 1$$

$$\Rightarrow \underbrace{-(x^2+y^2)^{1/2}}_{\xrightarrow{(x,y)\to(0,0)} 0} \leq (x^2+y^2)^{1/2} \sin\frac{1}{(x^2+y^2)} \leq \underbrace{(x^2+y^2)^{1/2}}_{\xrightarrow{(x,y)\to(0,0)} 0}$$

$$\Rightarrow (*) = 0$$

$\Rightarrow f$ is differentiable at the point zero.

Now let us compute the partial derivatives

$$(x,y) = (0,0) : f = 0 \Rightarrow \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$$

$$(x,y) \neq (0,0) : f = (x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right)$$

$$\frac{\partial f}{\partial x} = (2x)\sin\frac{1}{x^2+y^2} + (x^2+y^2) \cdot \frac{-2x}{(x^2+y^2)^2} \cdot \cos\left(\frac{1}{x^2+y^2}\right)$$

$$\frac{\partial f}{\partial y} = (2y)\sin\frac{1}{x^2+y^2} + (x^2+y^2) \cdot \frac{-2y}{(x^2+y^2)^2} \cdot \cos\left(\frac{1}{x^2+y^2}\right)$$

Therefore, partial derivatives are everywhere defined. Now, we check their continuity.

For $(x,y)\neq(0,0)$ $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are both continuous functions. we just need to check $(x,y)=(0,0)$

$$\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x} = \lim_{(x,y)\to(0,0)} \left[(2x)\sin\frac{1}{x^2+y^2} - \frac{2x\cancel{(x^2+y^2)}}{(x^2+y^2)^{\cancel{2}}}\cdot\cos\left(\frac{1}{x^2+y^2}\right)\right] \quad (**)$$

Let us choose the direction $x=y$

$$(**) = \lim_{x\to0}\left[\underbrace{2x\sin\left(\frac{1}{2x^2}\right)}_{\to 0} - \frac{\cancel{2x}}{(\cancel{2}x^{\cancel{2}})}\cdot\cos\left(\frac{1}{2x^2}\right)\right]$$

because $-1 \leq \sin\frac{1}{2x^2} \leq 1$

$$= -\lim_{x\to0}\frac{1}{x}\cos\frac{1}{2x^2}$$ this limit doesn't exist.

$\Rightarrow$ Although f is differentiable at $(0,0)$, the partial derivatives are not continuous at $(0,0)$. /

For problem 4 use remark 2 above and some

similar steps as Exercise 1 for the "problematic"

points.

For Problem 5, Remember:

$f: \mathbb{R} \to \mathbb{R}^3$ means $f(x) = (f_1(x), f_2(x), f_3(x))$

where each f_i is a real function $f_i: \mathbb{R} \to \mathbb{R}$

Therefore, $f(x)$ is a vector.

for any vector $v \in \mathbb{R}^n$, $|v| = (v_1^2 + \cdots + v_n^2)^{\frac{1}{2}}$

* $(f^n)' = n f^{n-1} f'$

* $f \cdot \nabla f$ is the inner product of two vectors.