HW 14. Solutions.

problem 1. $f(r,\theta,\phi) = (\underbrace{r\sin\theta\cos\phi}_{f_1}, \underbrace{r\sin\theta\sin\phi}_{f_2}, \underbrace{r\cos\theta}_{f_3})$

$$\nabla f = \begin{bmatrix} \frac{\partial f_1}{\partial r} & \frac{\partial f_1}{\partial \theta} & \frac{\partial f_1}{\partial \phi} \\ \frac{\partial f_2}{\partial r} & \frac{\partial f_2}{\partial \theta} & \frac{\partial f_2}{\partial \phi} \\ \frac{\partial f_3}{\partial r} & \frac{\partial f_3}{\partial \theta} & \frac{\partial f_3}{\partial \phi} \end{bmatrix}$$

$$= \begin{bmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{bmatrix}$$

This is the differential. Now we compute the Jacobian determinant.

$$\det \nabla f = \cos\theta\left(r^2\sin\theta\cos\theta\cos^2\phi + r^2\sin\theta\cos\theta\sin^2\phi\right)$$

↳ wrt the last row

$$+ r\sin\theta\left(r\sin^2\theta\cos^2\phi + r\sin^2\theta\sin^2\phi\right)$$

$$= r^2\sin\theta\cos^2\theta\,(\overbrace{\cos^2\phi + \sin^2\phi}^{1})$$

$$+ r^2 \sin^3\theta \, (\underbrace{\cos^2\phi + \sin^2\phi}_{1})$$

$$= r^2 \sin\theta \, (\underbrace{\cos^2\theta + \sin^2\theta}_{1}) = r^2 \sin\theta \, . \checkmark$$

problem 2. $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ differentiable
$(\alpha, \beta) \longmapsto f(\alpha, \beta)$

$g : \mathbb{R}^3 \longrightarrow \mathbb{R}$ such that

$$g(x, y, z) = f(x+y+z \, , \, x^2+y^2+z^2)$$

define $h : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ as

$$h(x, y, z) = (\underbrace{x+y+z}_{h_1} \, , \, \overbrace{x^2+y^2+z^2}^{h_2})$$

Lemma 4.5. says $\nabla g = \nabla f(h) \cdot \nabla h$

This means at a point (x, y, z)

$$\nabla g(x, y, z) = \begin{bmatrix} \frac{\partial f}{\partial \alpha}(h(x,y,z)) & \frac{\partial f}{\partial \beta}(h(x,y,z)) \end{bmatrix}_{1 \times 2}$$

$$\times \begin{bmatrix} \frac{\partial h_1}{\partial x}(x,y,z) & \frac{\partial h_1}{\partial y}(x,y,z) & \frac{\partial h_1}{\partial z}(x,y,z) \\ \frac{\partial h_2}{\partial x}(x,y,z) & \frac{\partial h_2}{\partial y}(x,y,z) & \frac{\partial h_2}{\partial z}(x,y,z) \end{bmatrix}_{2 \times 3}$$

you can not improve it more from here because you do not know what f is.

$$= \overbrace{\left[\frac{\partial f}{\partial \alpha}(h(x,y,z)) \quad \frac{\partial f}{\partial \beta}(h(x,y,z)) \right]} \times \begin{bmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \end{bmatrix} .$$

Extra: To have a better understanding, set an easy function for f. For example $f(\alpha,\beta) = \alpha^2 + 2\beta^2$

$$\frac{\partial f}{\partial \alpha} = 2\alpha \qquad \frac{\partial f}{\partial \beta} = 4\beta$$

$$\Rightarrow \nabla g(x,y,z) = \left[2(x+y+z) \quad 4(x^2+y^2+z^2) \right] \times \begin{bmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \end{bmatrix}$$

$$= \left[2(x+y+z) + 8x(x^2+y^2+z^2) \quad 2(x+y+z) + 8y(x^2+y^2+z^2) \quad 2(x+y+z) + 8z(x^2+y^2+z^2) \right]$$

note that $g = f \circ h = (x+y+z)^2 + 2(x^2+y^2+z^2)^2$

now directly compute ∇g, do you get the same?

You should. :)

problem 3. Using the theorem (Thm 20.1 in the link)

we need to show that $\exists U \subset \mathbb{R}^2$ such that U is

an open neighbourhood of $(1,1)$ and

1. $f \in C^1(U, \mathbb{R}^2)$ i.e. f is differentiable on U

and the differential is continuous.

2. $\det f'((1,1)) \neq 0$

$f : \mathbb{R}^2 \to \mathbb{R}^2$ $\quad f(x,y) = (\overbrace{2xy}^{f_1}, \overbrace{e^x + y}^{f_2})$

$\frac{\partial f_1}{\partial x} = 2y \qquad \frac{\partial f_1}{\partial y} = 2x$

$\frac{\partial f_2}{\partial x} = e^x \qquad \frac{\partial f_2}{\partial y} = 1$

all the partial derivatives are continuous on all $\mathbb{R}^2$

$\Rightarrow f$ is differentiable and $\nabla f = \begin{bmatrix} 2y & 2x \\ e^x & 1 \end{bmatrix}$

Note that $\nabla f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$(x,y) \longmapsto \begin{bmatrix} 2y & 2x \\ e^x & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{(4xy \quad xe^x + y)}_{\text{both are continuous everywhere.}}$$

$\Rightarrow \nabla f$ is a continuous map on $\mathbb{R}^2$.

$$\det \nabla f = 2y - 2xe^x$$

evaluating at $(1,1)$: $2(1) - 2(1)e^1 = 2 - 2e \neq 0$

$\Rightarrow \exists \underset{\text{open}}{U_0} \subset \mathbb{R}^2$ such that $(1,1) \in U_0$

and f is invertible on U_0.

problem 4. $f : \mathbb{R}^d \longrightarrow \mathbb{R}^d \qquad f(x) = x|x|$

First we start with computing the partial derivatives and see for which $(x_1, \dots, x_d) \in \mathbb{R}^d$, they are continuous. Then we know that for those points f is differentiable.

Let $x = (x_1, \dots, x_d)$ $\quad f(x_1, \dots, x_d) = |x|(x_1, \dots, x_d)$

$$= (x_1^2 + \dots + x_d^2)^{\frac{1}{2}} (x_1, \dots, x_d)$$

$$= \Big(\underbrace{(x_1^2 + \dots + x_d^2)^{\frac{1}{2}} x_1}_{f_1}, \underbrace{(x_1^2 + \dots + x_d^2)^{\frac{1}{2}} x_2}_{f_2}, \dots, \underbrace{(x_1^2 + x_2^2 + \dots + x_d^2)^{\frac{1}{2}} x_d}_{f_d} \Big)$$

$$\frac{\partial f_1}{\partial x_1} = \frac{1}{2} (x_1^2 + \dots + x_d^2)^{-\frac{1}{2}} (2x_1)(x_1) + (x_1^2 + \dots + x_d^2)^{\frac{1}{2}}$$

This is a continuous function everywhere but

the point $(x_1, \dots, x_d) = (0, \dots, 0)$.

Similarly you can compute $\frac{\partial f_i}{\partial x_j}$ $i, j \in \{1, \dots, d\}$ and

they are all continuous everywhere but at 0.

Therefore f is differentiable on all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$

Now we need to check the differentiability at 0.

Remember: No continuous partial derivative does

not imply no differential. See Exercise 5.

Similar to Exercise 1, we use the original definition

If we find a linear function $L : \mathbb{R}^d \to \mathbb{R}$ such

that

$$(*) \quad \lim_{(x_1,\dots,x_d)\to(0,\dots,0)} \frac{f(x_1,\dots,x_d) - f(0,\dots,0) - L((x_1,\dots,x_d)-(0,\dots,0))}{\|(x_1,\dots,x_d)\|}$$

is zero then f is differentiable at 0.

$X = (x_1,\dots,x_d)$

$$(*) = \lim_{X\to 0} \frac{f(X) - L(X)}{|X|} = \lim_{X\to 0} \frac{X|X| - L(X)}{|X|}$$

Let $L(X) = 0$ then $\lim_{X\to 0} \frac{X|X|}{|X|} = \lim_{X\to 0} X = 0$

$\Rightarrow f$ is differentiable everywhere on $\mathbb{R}^2$.

problem 5. $\quad f : \mathbb{R} \longrightarrow \mathbb{R}^3$

$$x \longmapsto (f_1(x), f_2(x), f_3(x))$$

s.t. $f_i : \mathbb{R} \to \mathbb{R} \quad i \in \{1,2,3\}$

$$1 = |f(x)| = \left(f_1^2(x) + f_2^2(x) + f_3^2(x) \right)^{1/2}$$

$$\Rightarrow f_1^2(x) + f_2^2(x) + f_3^2(x) = 1$$

f is differentiable therefore f_1, f_2, f_3 are differentiable

now take the derivative wrt x from both sides.

$$\Rightarrow 2f_1 f_1' + 2f_2 f_2' + 2f_3 f_3' = 0$$

$$\Rightarrow f_1 f_1' + f_2 f_2' + f_3 f_3' = 0$$

$$\Rightarrow \begin{pmatrix} f_1 & f_2 & f_3 \end{pmatrix} \begin{bmatrix} \frac{\partial f_1}{\partial x} \\ \frac{\partial f_2}{\partial x} \\ \frac{\partial f_3}{\partial x} \end{bmatrix} = 0$$

$$\Rightarrow f \cdot \nabla f = 0 \quad \checkmark$$