Problem Sheet 11

Due Date: 25.05.2020, 12:00 UTC+2 (CEST)

Problem 1. [**2+2** pts] For the following linear mappings φ find basis and dimension of $ker\ \varphi$ and $im\ \varphi$

- $\varphi : \mathbb{R}^3 \to \mathbb{R}^4, \quad \varphi(x,y,z) = (4x+3y+5z, x+2y+z, 2x-y+3z, 6x+7y+7z)$
- $\varphi : \mathbb{R}^4 \to \mathbb{R}^3$,
  $\varphi(x_1,x_2,x_3,x_4) = (5x_1+6x_2+4x_3+7x_4, x_1+3x_2+2x_3+4x_4, 7x_1+3x_2+2x_3+2x_4)$

Problem 2. [**2+2** pts] Find eigenpairs of the following tensors

- $S : \mathbb{R}^3 \to \mathbb{R}^3, \quad S(x,y,z) = (x-y, x+3y+z, 2z)$
- $S : \mathbb{R}^3 \to \mathbb{R}^3, \quad S(x,y,z) = (4x+y, 3x+2y, 7x-7y+5z)$

Problem 3. [**3** pts] Let $\{a^1, \dots a^d\}$ be a basis of a linear space V. Consider the tensor $S : V \to V$ such that

$$Sa^i = a^{i+1} \quad \text{for } i = 1, \dots d-1, \qquad Sa^d = \alpha_i a^i \quad \text{for certain } \alpha_i \in \mathbb{R},\ i = 1, \dots d$$

what is the characteristic polynomial of S?

Problem 4. [**4** pts] Consider a tensor $S : \mathbb{R}^3 \to \mathbb{R}^3$ represented in the standard basis by the following matrix

$$\begin{bmatrix} 1 & \sqrt{3} & 1 \\ \sqrt{3} & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Compute S^{101}, $exp(S)$. How many square roots has S?

Problem 5. [**3** pts] Show that the principal invariants I_k of $S : \mathbb{R}^3 \to \mathbb{R}^3$ satisfy for any orthogonal tensor Q

$$I_k(QSQ^T) = I_k(S)$$

(remark: that is why they are called 'invariants')

Problem 6. [**4** pts] Consider SOLE

$$\begin{cases} 3x_1 + x_2 + 3x_3 = 2 \\ 4x_1 + 4x_2 + 7x_3 = t \\ sx_2 + 5x_1 + 11x_3 = 0 \\ 2x_1 - 2x_2 - x_3 = 0 \end{cases} \tag{1}$$

Find values of the parameters $s, t \in \mathbb{R}$ such that the SOLE has: (i) none, (ii) one, (iii) many solutions.

Problem 7. [**3** pts] Let $T : V \to W$ be a tensor such that its matrix in the bases f^1, f^2, f^3 of V and g^1, g^2 of W is

$$\begin{bmatrix} 0 & -1 & 2 \\ 3 & -4 & 5 \end{bmatrix}$$

Find the matrix representation of T in the bases: $f^1, f^1+f^2, f^1+f^2-f^3$ of V and g^1, g^1-g^2 of W.