Solutions to Problem sheet 2

Let V be a finite-dim vector space with two different bases $\{v_1, \ldots, v_n\}$ and $\{v'_1, \ldots, v'_n\}$. Let the basis change be denoted by $v'_i = \sum_{j=1}^{n} C^j_i v_j$ $\quad C^{j \to \text{row}}_{i \to \text{column}}$

ⓐ Let $(v^*_i \otimes v^*_j)_{1 \leq i,j \leq n}$ denote the canonically associated basis of $V^* \otimes V^*$, where $\quad v^*_i \otimes v^*_j (v_k, v_\ell) = \delta_{ik} \delta_{j\ell}$.

Suppose $\beta \in V^* \otimes V^*$ is expressed ~~[illegible]~~ by:

$$\beta = \sum_{i,j} B_{ij} v^*_i \otimes v^*_j = \sum_{i,j} B'_{ij} (v'^*_i \otimes v'^*_j)$$

How B and B' are related? (use C, C^T, C^{-1})

$\beta \in V^* \otimes V^*$ is a bilinear form $\beta : V \times V \longrightarrow \mathbb{R}$.

$$\beta(\lambda'_1 v'_1 + \cdots + \lambda'_n v'_n \,,\, \mu'_1 v'_1 + \cdots + \mu'_n v'_n) = \sum_{i,j} \lambda'_i \mu'_j B'_{ij}$$

$$= (\lambda'_1, \ldots, \lambda'_n) B' \begin{pmatrix} \mu'_1 \\ \vdots \\ \mu'_n \end{pmatrix} \quad (*)$$

$$v'_i = \sum_{j=1}^{n} C^j_i v_j \Rightarrow \sum_{i=1}^{n} \mu'_i v'_i = \sum_{i=1}^{n} \mu'_i \left(\sum_{j=1}^{n} C^j_i v_j \right)$$

$$= \sum_{j=1}^{n} \underbrace{\left(\sum_{i=1}^{n} \mu'_i C^j_i \right)}_{\mu_j} v_j$$

So we have $\begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix} = C \begin{pmatrix} \mu'_1 \\ \vdots \\ \mu'_n \end{pmatrix} \Rightarrow \begin{pmatrix} \mu'_1 \\ \vdots \\ \mu'_n \end{pmatrix} = C^{-1} \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}$

Similarly you can show $(\lambda'_1, \ldots, \lambda'_n) = (\lambda_1, \ldots, \lambda_n)(C^{-1})^T$

Substituting new representations of λ' and μ' in $(*)$

gives

$$(\lambda_1, \dots, \lambda_n) \underbrace{(C^{-1})^T B' (C^{-1})}_{B} \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}$$

$$\text{So} \quad B = (C^{-1})^T B' (C^{-1})$$

$$B' = C^T B C$$

Now for $h \in V \otimes V$, use the fact that

$$\begin{tikzcd} \{v_1', \dots, v_n'\} \arrow[d, "C"'] & \{v_1'^*, \dots, v_n'^*\} \\ \{v_1, \dots, v_n\} & \{v_1^*, \dots, v_n^*\} \arrow[u, "C^T"'] \end{tikzcd}$$

Thus,

$$h = \sum_{ij} H^{ij} v_i \otimes v_j = \sum H'^{ij} v_i' \otimes v_j'$$

$$\Rightarrow H' = ((C^{-1})^T)^T H ((C^{-1})^T) = C^{-1} H (C^{-1})^T$$

(you simply exchang C with $(C^{-1})^T$.

ⓑ $T \in \underbrace{V \otimes \cdots \otimes V}_{r} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{s}$,

$$T = \sum_{\substack{i_1, \dots, i_r \\ j_1, \dots, j_s}} T^{i_1, \dots, i_r}_{j_1, \dots, j_s} (v_{i_1} \otimes \cdots \otimes v_{i_r} \otimes v^{*j_1} \otimes \cdots \otimes v^{*j_s}) \qquad (*)$$

$$= \sum_{\substack{i_1, \dots, i_r \\ j_1, \dots, j_s}} T'^{i_1, \dots, i_r}_{j_1, \dots, j_s} (v'_{i_1} \otimes \cdots \otimes v'_{i_r} \otimes v'^{*j_1} \otimes \cdots \otimes v'^{*j_s})$$

we want to find the relation between T and T'.

$$\sum_{\substack{i_1,\dots,i_r\\ j_1,\dots,j_s}} T'^{\,i_1,\dots,i_r}_{\;j_1,\dots,j_s} \left(v'_{i_1}\otimes\cdots\otimes v'_{i_r}\otimes v'^{*j_1}\otimes\cdots\otimes v'^{*j_s}\right) =$$

$$\sum_{\substack{i_1,\dots,i_r\\ j_1,\dots,j_s}} T'^{\,i_1,\dots,i_r}_{\;j_1,\dots,j_s} \Bigg(\sum_{k_1=1}^{n} C^{k_1}_{i_1} v_{k_1} \otimes\cdots\otimes \sum_{k_r=1}^{n} C^{k_r}_{i_r} v_{k_r}$$

$$\otimes \sum_{l_1=1}^{n} \left(\left(C^{-1}\right)^T\right)^{l_1}_{j_1} v^{*l_1} \otimes\cdots\otimes \sum_{l_s=1}^{n} \left(\left(C^{-1}\right)^T\right)^{l_s}_{j_s} v^{*l_s} \Bigg) =$$

$$\sum_{\substack{i_1,\dots,i_r\\ j_1,\dots,j_s}} T'^{\,i_1,\dots,i_r}_{\;j_1,\dots,j_s} \sum_{\substack{k_1,\dots,k_r,\\ l_1,\dots,l_s}} C^{k_1}_{i_1}\cdots C^{k_r}_{i_r} \left(C^{-1}\right)^{T\,l_1}_{j_1}\cdots\left(C^{-1}\right)^{T\,l_s}_{j_s} \left(v_{k_1}\otimes\cdots\otimes v_{k_r} \otimes v^{*l_1}\otimes\cdots\otimes v^{*l_s}\right)$$

$$= \sum_{\substack{k_1,\dots,k_r\\ l_1,\dots,l_s}} \underbrace{\sum_{\substack{i_1,\dots,i_r\\ j_1,\dots,j_s}} T'^{\,i_1,\dots,i_r}_{\;j_1,\dots,j_s} C^{k_1}_{i_1}\cdots C^{k_r}_{i_r} \overset{(C^{-1})^{j_1}_{l_1}}{\left(C^{-1}\right)^{T\,l_1}_{j_1}}\cdots\overset{(C^{-1})^{j_s}_{l_s}}{\left(C^{-1}\right)^{T\,l_s}_{j_s}}}_{\overset{\parallel}{T^{k_1,\dots,k_r}_{l_1,\dots,l_s}} \;\longleftarrow\; \text{from } (*)} \left(v_{k_1}\otimes\cdots\otimes v_{k_r}\otimes v^{*l_1}\otimes\cdots\otimes v^{*l_s}\right)$$

Therefore,

$$T^{k_1,\dots,k_r}_{l_1,\dots,l_s} = \sum_{\substack{i_1,\dots,i_r\\ j_1,\dots,j_s}} T'^{\,i_1,\dots,i_r}_{\;j_1,\dots,j_s} C^{k_1}_{i_1}\cdots C^{k_r}_{i_r} \left(C^{-1}\right)^{j_1}_{l_1}\cdots\left(C^{-1}\right)^{j_s}_{l_s}$$

for $k_a, l_b, i_a, j_b \in \{1,\dots,n\}$ and $1\le a\le r$, $1\le b\le s$

(Note: you can replace T with T' by replacing C with C^{-1}.)

2.-(a) $(x+y)\operatorname{Cosh}(x-y) = 2x$ (we use implicit function theorem)

let $F: \overset{X}{\mathbb{R}} \times \overset{= Y}{R} \longrightarrow \overset{Z}{\mathbb{R}}$

$(x, y) \longmapsto (x+y)\operatorname{Cosh}(x-y) - 2x$

$$\operatorname{Cosh} u = \frac{e^{u} + e^{-u}}{2}$$

let $x_0 = 1$ U = R x R $F_{x_0} : \{y \in \mathbb{R} : (x_0, y) \in \mathbb{R}\} \longrightarrow \mathbb{R}$

$y \longmapsto (1+y)\operatorname{Cosh}(1-y) - 2$

let $y_0 = 1$ $(x_0, y_0) \in U$ ✓

$$\frac{\partial F}{\partial y}(x_0, y_0) := DF_{x_0}(y_0) = \operatorname{Cosh}(1-y) - (1+y)\left.\frac{e^{1-y} - e^{y-1}}{2}\right|_{y_0=1} = 1$$

This is invertible

$F(x_0, y_0) = F(1,1) = 0$

So by implicit function theorem $\exists U_{x_0} \overset{x_0 \subset U_{x_0}}{\subset} X$ and $h \in C^1(U_{x_0}, Y)$

s.t. $F(x, h(x)) = 0 \quad \forall x \in U_{x_0}$.

Note that for sufficiently small ε $(1-\varepsilon, 1+\varepsilon) \subset U_{x_0}$

$F(1,1) = 0 \Rightarrow h(1) = 1$

$$Dh(x_0) = -\underbrace{\left(\frac{\partial F}{\partial y}(x_0, y_0)\right)^{-1}}_{1} \cdot \underbrace{\left(\frac{\partial F}{\partial x}(x_0, y_0)\right)}_{\hookleftarrow}$$

$$\frac{\partial F}{\partial x}(x_0, y_0) = \cancel{\frac{\partial F}{\partial x}}\, D\left(x \mapsto F(x, y_0)\right)(x_0) =$$

$$D\left(x \mapsto (x+1)\operatorname{Cosh}(x-1) - 2x\right)(x_0) = \operatorname{Cosh}(x-1) + (x+1)\left.\frac{e^{x-1} - e^{1-x}}{2} - 2\right|_{x_0=1}$$

$= 1 - 2 = -1$

$Dh(x_0) = -(1)^{-1}(-1) = 1$

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$M = \mathcal{M}(n\times n, \mathbb{C}) \cong \mathbb{R}^{2n^2}$

2-(b) Let $f: M \to S := \{B \in M : B = \bar{B}^T\}$
$A \mapsto A.\bar{A}^T$ (simply check that $A\bar{A}^T \in S, \forall A \in M$)

Let $F: M \to S$ i.e., $F(A) = f(\mathbb{1}+A)$
$A \mapsto f(\mathbb{1}+A)$

$DF(0) = Df(\mathbb{1})$ and $Df(\mathbb{1}) : M \to S$
$B \mapsto B + \bar{B}^T$ } → why (?)

we show it at the end of the proof.

Now it is easy to see that $Df(\mathbb{1})$ is onto bcz any $C \in S$ can be written as $\frac{C}{2} + \frac{C}{2} = \frac{C}{2} + \frac{\bar{C}^T}{2} = Df(\mathbb{1})(\frac{C}{2})$

$C = \bar{C}^T$ $(C \in S)$

Define $X := \ker Df(\mathbb{1}) \Rightarrow X = \{B \in M : B = -\bar{B}^T\} = \{B \in M : B^T = -\bar{B}\}$

let the (i,j) entry of B, be denoted by $a_{ij} + i b_{ij}$, by $B^T = -\bar{B}$

we should have $a_{ii} = 0$ $\forall 1 \leq i \leq n$, $a_{ij} = -a_{ji}$ $\forall i,j$, $b_{ji} = b_{ij}$ $\forall i \neq j$

and b_{ii} free for all $1 \leq i \leq n$. So $\dim X = \underbrace{\frac{n^2-n}{2}}_{\text{for } a} + \underbrace{\frac{n^2-n}{2} + n}_{\text{for } b}$

$\rightarrow \boxed{\dim X = n^2}$

let $Y \subset M$ be a subset of M of dim $2n^2 - n^2 = n^2$ such that $X \cap Y = 0$ and $M = X \oplus Y$

$F(0) = f(\mathbb{1}) = \mathbb{1}$

Now Implicit function theorem implies that $\exists U \subset X$ s.t. $0 \subset U$

and $\exists h : U \to Y$ s.t. $F(A, h(A)) = \mathbb{1}$ $\forall A \in U$

$$\underset{\text{near } \mathbb{1}}{U(n,\mathbb{C})} = \underset{\text{near } \mathbb{1}}{f^{-1}(\mathbb{1})} = \mathbb{1} + \underbrace{F^{-1}(\mathbb{1})}_{\text{near } 0}$$

So near the identity $\mathbb{1}$ $U(n,\mathbb{C})$ can be parametrized by an open subset of $\mathbb{1} + X$ because near 0 $F^{-1}(\mathbb{1})$ can be parametrized by an open subset of $X = \{B \in M \mid B^T = -\bar{B}\}$ and we calculated that "$\dim \mathbb{1} + X = \dim X = n^2$" ✓

*

Let the conjugate transpose of a matrix M be M*:

$$(DF(A))(B) = \frac{d}{dt} F(A+tB)\Big|_{t=0} = \frac{d}{dt} (A+tB)(A+tB)^*\Big|_{t=0} = \frac{d}{dt} (A+tB)(A^*+tB^*)\Big|_{t=0} =$$

$$\frac{d}{dt}(AA^*+t(AB^*+BA^*)+t^2 BB^*)\Big|_{t=0} = AB^*+BA^*$$

Now set A = I.

Pages 35 and 36 are removed.

③ ⓐ WTS $f : \mathbb{CP}^n \longrightarrow X$ continuous $\Longleftrightarrow$ $f \circ \pi : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow X$ continuous.

WE KNOW:

$\pi : \mathbb{C}^{n+1} \setminus \{0\} \longrightarrow \mathbb{CP}^n$

$(z_0, \dots, z_n) \longmapsto [z_0 : \dots : z_n]$ → equivalence class

BY DEF: $\mathbb{CP}^n = \dfrac{\mathbb{C}^{n+1} \setminus \{0\}}{\sim}$ where $(z_0, \dots, z_n) \sim (w_0, \dots, w_n)$

$\Updownarrow$

$\exists \lambda \overset{\neq 0}{\in} \mathbb{C}$ s.t. $(z_0, \dots, z_n) = \lambda (w_0, \dots, w_n)$

Thus $[1:0:2] = [4:0:8] = [i+2:0:2i+4] = \cdots$

BY DEF: $U \subset \mathbb{CP}^n$ is open $\Longleftrightarrow$ $\pi^{-1}(U) \subset \mathbb{C}^{n+1} \setminus \{0\}$ is open

let $U \subset X$ open: $f^{-1}(U) \subseteq \mathbb{CP}^2$ open $\Longleftrightarrow$ $\pi^{-1}(f^{-1}(U))$ open in $\mathbb{C}^{n+1} \setminus \{0\}$ $\Longleftrightarrow$ $(f \circ \pi)^{-1}(U)$ open in $\mathbb{C}^{n+1} \setminus \{0\}$. /

$g : X \longrightarrow \mathbb{CP}^n$ let $U \subseteq \mathbb{CP}^n$ open

$V \subset X$ open $\Longleftrightarrow$ $\forall x \in V$ $\exists$ nbhd V_x of x s.t. $V_x \subset V$

Thm 4 in my note for basic topology, also presented in your lectures.

$g^{-1}(U)$ open in X $\Longleftrightarrow$ $\forall x \in g^{-1}(U)$ $\exists$ nbhd V_x of x s.t. $V_x \in g^{-1}(U)$

$\Longleftrightarrow$ $\forall x$ s.t. $g(x) \in U$ $\exists$ nbhd V_x of x s.t. $g(V_x) \subset U$.

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ⓑ $U_i \subset \mathbb{CP}^n$ $U_i := \{[z_0, \dots, z_n] \mid z_i \neq 0\}$ with subset top.

WTS $\varphi_i : \mathbb{C}^n \longrightarrow U_i$ are homeom. $\forall i \in \{0, \dots, n\}$

$(w_1, \dots, w_n) \longmapsto [w_1 : \dots : w_i : 1 : w_{i+1} : \dots : w_n]$

i.e. we need to check that φ_i is <u>bijection</u>, <u>continuous</u> and it has a <u>continuous inverse</u>.

$\boxed{\text{bijection}}$: Let $\psi_i : U_i \to \mathbb{C}^n$

$$[z_0 : \dots : z_n] \longmapsto \left(\frac{z_0}{z_i}, \dots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \dots, \frac{z_n}{z_i}\right)$$

You can easily check that $\psi_i \circ \varphi_i = \mathbb{1}_{\mathbb{C}^n}$ and $\varphi_i \circ \psi_i = \mathbb{1}_{U_i}$. ✓

$\boxed{\text{Continuous}}$: WTS $\forall V \subset U_i$ open and $\forall x \in \mathbb{C}^n$ s.t. $\varphi_i(x) \in V$

$\exists W \subset \mathbb{C}^n$ open s.t. $x \in W$ and $\varphi_i(W) \subset V$.

$V \subset U_i$ open and $\overbrace{U_i \subset \mathbb{CP}^n \text{ open}}^{*}$ $\Rightarrow$ V open in $\mathbb{CP}^n$

$\overset{\text{by def.}}{\Longrightarrow}$ $\pi^{-1}(V)$ open in $\mathbb{C}^{n+1} \setminus \{0\}$.

$\pi^{-1}(U_i) = \{ \overset{(x_0, \dots, x_n)}{x} \in \mathbb{C}^{n+1} \setminus \{0\} \mid x_i \neq 0 \}$

$F_i : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}$

$(x_1, \dots, x_n) \mapsto x_i$

$\pi^{-1}(U_i) = \underbrace{\mathbb{C}^{n+1} \setminus \overbrace{F_i^{-1}(0)}^{\text{clsd}}}_{\text{open in } \mathbb{C}^{n+1} \setminus \{0\}}$

Define a new map: $\tilde{\varphi}_i : \mathbb{C}^n \longrightarrow \mathbb{C}^{n+1} \setminus \{0\}$

$(w_1, \dots, w_n) \longmapsto (w_1, \dots, w_i, 1, w_{i+1}, \dots, w_n)$

Now you can easily check that $\forall x \in \mathbb{C}^n$:

$$\varphi_i(x) = \pi \circ \tilde{\varphi}_i(x) \Rightarrow \left(\begin{array}{l} \varphi_i(x) \in V \Longleftrightarrow \pi \circ \tilde{\varphi}_i(x) \in V \\ \qquad\qquad \Longleftrightarrow \tilde{\varphi}_i(x) \in \pi^{-1}(V) \end{array} \right)$$

Since $\tilde{\varphi}_i$ is continuous, $\exists$ nbhd $W \subset \mathbb{C}^n$ of x s.t. $\tilde{\varphi}_i(W) \subset \pi^{-1}(V)$

$\Rightarrow \pi \circ \tilde{\varphi}_i(W) \subset V \Rightarrow \varphi_i(W) \subset V$. ✓

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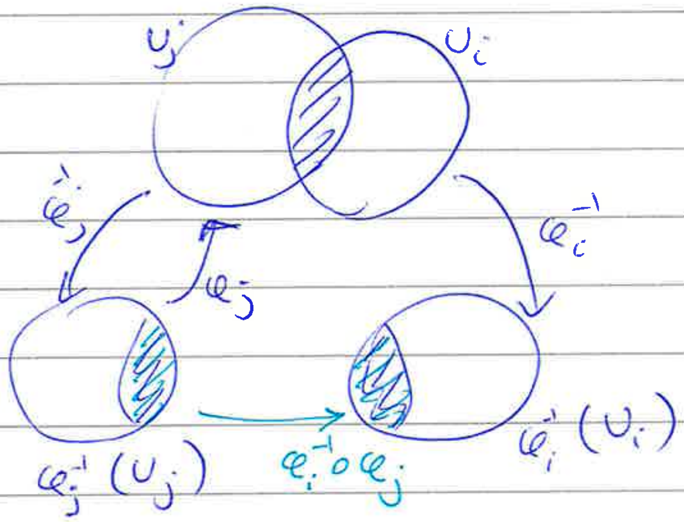
$\boxed{\text{Continuous inverse}}$ WTS ψ_i is cont.

$$\psi_i \circ \pi\big|_{\pi^{-1}(U_i)} : \pi^{-1}(U_i) \longrightarrow \mathbb{C}^n$$

$$(z_0, \dots, z_n) \longmapsto \left(\frac{z_0}{z_i}, \dots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \dots, \frac{z_n}{z_i}\right)$$

is cont. $\xRightarrow{(a)}$ ψ_i is cont. /

ⓒ we want to compute $\varphi_i^{-1} \circ \varphi_j$



$0 \le i, j \le n$

$$\varphi_i^{-1} \circ \varphi_j \Big|_{\varphi_j^{-1}(U_i \cap U_j)} : \varphi_j^{-1}(U_i \cap U_j) \longrightarrow \varphi_i^{-1}(U_i \cap U_j)$$

let $(z_1, \dots, z_n) \in \varphi_j^{-1}(U_i \cap U_j) \subseteq \mathbb{C}^n$

$$\varphi_i^{-1} \circ \varphi_j (z_1, \dots, z_n) = \varphi_i^{-1}([z_1 : \dots : z_j : 1 : z_{j+1} : \dots : z_n])$$

$$= \begin{cases} \left(\frac{z_1}{z_{i+1}}, \dots, \frac{z_i}{z_{i+1}}, \frac{z_{i+2}}{z_{i+1}}, \dots, \frac{z_j}{z_{i+1}}, \frac{1}{z_{i+1}}, \frac{z_{j+1}}{z_{i+1}}, \dots, \frac{z_n}{z_{i+1}}\right) & i < j-1 \\ \left(\frac{z_1}{z_i}, \dots, \frac{z_j}{z_i}, \frac{1}{z_i}, \frac{z_{j+2}}{z_i}, \dots, \frac{z_n}{z_i}\right) & i = j+1 \\ \left(\frac{z_1}{z_i}, \dots, \frac{z_j}{z_i}, \frac{1}{z_i}, \frac{z_{j+1}}{z_i}, \dots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \dots, \frac{z_n}{z_i}\right) & i > j+1 \end{cases}$$

$$\left(\frac{z_1}{z_{i+1}}, \dots, \frac{z_{j-1}}{z_{i+1}}, 1, \frac{z_{j+1}}{z_{i+1}}, \dots, \frac{z_n}{z_{i+1}}\right) \leftarrow i = j-1$$

/ Z i+1

the collection $\{(U_i, \varphi_i^{-1}, \mathbb{C}^n) \mid i=0,\dots,n\}$ defines a smooth atlas on $\mathbb{CP}^n$ becaus for any $x \in \mathbb{CP}^n$, $\exists U_i$ for some $i=0,\dots,n$ s.t. $x \in U_i$ and $\varphi_i^{-1} : U_i \to \mathbb{C}^n$ is a homeomorphism. /

ⓓ from ⓐ-ⓒ and the fact that all transition functions $\varphi_i^{-1} \circ \varphi_j$ are smooth (i.e. they have continuous partial derivatives of all orders). $\mathbb{CP}^n$ is a smooth manifold.

NOTE: It is not uncommon to demand in the definition of a manifold that it be second-countable and Hausdorff. (this is actually the case in your Lecture notes, so we need to show that $\mathbb{CP}^n$ is Hausdorff and second countable.)

$\boxed{\text{Hausdorff}}$ Let $x, y \in \mathbb{CP}^n$ s.t. $x \neq y$. WTS $\exists U_x, U_y \underset{\text{open}}{\subset} \mathbb{CP}^n$ s.t. $U_x \cap U_y = \emptyset$. By the definition of π, if $x \neq y \Rightarrow \pi^{-1}(x) \neq \pi^{-1}(y)$. $\mathbb{C}^{n+1} \setminus \{0\}$ is Hausdorff so $\exists V_x, V_y \underset{\text{open}}{\subset} \mathbb{C}^{n+1} \setminus \{0\}$ s.t. $\pi^{-1}(x) \subset V_x$ and $\pi^{-1}(y) \subset V_y$ and $V_x \cap V_y = \emptyset$. Let $U_x = \pi^{-1}(V_x)$ and $U_y = \pi^{-1}(V_y)$. /

$\boxed{\text{Second countable}}$. It comes from the fact that $\mathbb{C}^{n+1} \setminus \{0\}$ is Second countable. A countable basis for $\mathbb{C}^{n+1}$ is

$\mathcal{B} = \{B_\varepsilon(a+bi)\}$ where $\overset{a=(a_0,\dots,a_n)\ \ b=(b_0,\dots,b_n)}{\varepsilon, a_i, b_i \in \mathbb{Q}}$ and $B_\varepsilon(a+bi)$ is the ball centered at $(a+bi) \subset \mathbb{C}^{n+1}$.

↳ rationals.

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So $\mathcal{B}' = \{B_{\epsilon}(a+bi) \cap \mathbb{C}^{n+1} \setminus \{0\}\}_{\epsilon,a,b}$ is a countable basis for $\mathbb{C}^{n+1} \setminus \{0\}$

and $\mathcal{B}'' = \{\pi(B_{\epsilon}(a+bi) \cap \mathbb{C}^{n+1} \setminus \{0\})\}$ is a countable basis for $\mathbb{CP}^n$.

Therefore, $\mathbb{CP}^n$ is a smooth manifold of complex dim n and real dim 2n.

∎

④ Lets review some definitions and notations first.

$U \subseteq \mathbb{R}^n$ open

* $\underline{\underline{C^\infty(U)}} = C^\infty(U, \mathbb{R}) = \{f : U \to \mathbb{R} \text{ s.t. } f \text{ is smooth on } U\}$

* A linear operator $X : C^\infty(U) \to C^\infty(U)$ is called a derivation or vector fields on U if:

$$(X(fg))(p) = f(p) \cdot (X(g))(p) + g(p) \cdot (X(f))(p)$$

$\forall f, g \in C^\infty(U)$ and $\forall p \in U$

$X(fg) = fXg + gXf$

to remember ;)

Denote the set of all vector fields on U by $\underline{\underline{\mathfrak{X}(U)}}$.

* $\mathfrak{X}(U)$ is a $\mathbb{R}$-vector space with addition and multiplication;

$(X_1 + X_2)(f) := X_1(f) + X_2(f)$ $\quad (rX)(f) = r(Xf)$ $\quad f \in C^\infty(U)$, $r \in \mathbb{R}$, $X_1, X_2, X \in \mathfrak{X}(U)$

proof: $\mathfrak{X}(U)$ is a subset of the $\mathbb{R}$-vector space $\mathrm{Hom}(C^\infty(U), C^\infty(U))$. you need to check that this subset is clsd under addition and multiplication. i.e. check that ① $(X_1 + X_2)(fg) = f(X_1+X_2)(g) + g(X_1+X_2)f$ and ② $(rX)(fg) = f(rX)(g) + g(rX)(f)$.

① $(X_1+X_2)(fg) = X_1(fg) + X_2(fg) = fX_1(g) + gX_1(f) + fX_2(g) + gX_2(f)$

$= f(X_1(g) + X_2(g)) + g(X_1(f) + X_2(f)) = f(X_1+X_2)(g) + g(X_1+X_2)(f)$.

② $(rX)(fg) = r\Big(X(fg)\Big) = r\Big(f\,Xg + gXf\Big) = f.(r.X(g)) + g.(r.X(f)) =$

$f\Big((rX)(g)\Big) + g\Big((rX)f\Big)$. ✓ /

The operator $[\cdot\,,\cdot] : \mathfrak{X}(U) \times \mathfrak{X}(U) \longrightarrow \mathfrak{X}(U)$

$$(X\,,\,Y) \longmapsto [X,Y] = \underbrace{X.Y - Y.X}_{\text{composition}}$$

is called the Lie bracket.

NOTE: Composition of two vector field is not necessarily a vector field but the Lie bracket is a vector field. Check it that

if $X, Y \in \mathfrak{X}(U)$ then XY is not always in $\mathfrak{X}(U)$ i.e.

$XY(fg) \neq f(XY)(g) + g(XY)f$. Lets check that $[X,Y] \in \mathfrak{X}(U)$

$$[X,Y](fg) = XY(fg) - YX(fg) = X\Big(fY(g) + gY(f)\Big) - Y\Big(fX(g) + gX(f)\Big)$$

$$= X\big(fY(g)\big) + X\big(gY(f)\big) - \Big[Y\big(fX(g)\big) + Y\big(gX(f)\big)\Big] =$$

$$fXY(g) + \cancel{Y(g)X(f)} + gX(Y(f)) + \cancel{Y(f)X(g)} - \Big[f(YX)(g) + \cancel{X(g)Y(f)} + gYX(f) + \cancel{X(f)Y(g)}\Big]$$

$$= f\Big((XY)(g) - (YX)(g)\Big) + g\Big((XY)(f) - (YX)(f)\Big) = f[X,Y](g) + g[X,Y](f)$$

✓/

So now the operator $[\cdot\,,\cdot]$ makes sense. :)

– There is an isomorphism $\mathfrak{X}(U) \cong C^\infty(U, \mathbb{R}^{n})$

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Remember that any vector space $X \in \mathfrak{X}(U)$ can be written uniquely of the form $X = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x_i}$ where $a^i \in C^\infty(U)$.

The isomorphism $\mathfrak{X}(U) \longrightarrow C^\infty(U, \mathbb{R}^n)$ is defined by maping X to the n-tuple $(a^1, \dots, a^n)$. so by $X(\varphi)$ (in the exercise 4) we mean $(a^1(p), \dots, a^n(p)) \in \mathbb{R}^n$

ⓐ WTS $df(x)=0 \Rightarrow X(Y(f))(x) = Y(X(f))(x)$

This is equivalent to say $df(x)=0 \Rightarrow ([X,Y]f)(x) = 0$

But we know that $df(x)=0 \iff (X(f))(x)=0 \quad \forall X \in \mathfrak{X}(U)$

(pf: because $X = \sum a^i \frac{\partial}{\partial x_i}$ and if $df(x)=0$ then $\frac{\partial f}{\partial x_i}(x) = 0$

$\Rightarrow X(f)(x)=0$ and if $(\sum a^i \frac{\partial f}{\partial x_i})(x) = 0 \quad \forall a^i \in C^\infty(U)$ then

$\forall_i \frac{\partial f}{\partial x_i} = 0 \Rightarrow df(x) = 0$)

Now since we proved that $[X,Y] \subset \mathfrak{X}(U)$, if $df(x)=0$ then $([X,Y]f)(x)=0 \Rightarrow ((XY)(f))(x) = ((YX)(f))(x)$ ✓

Now WTS that $((XY)(f))(x)$ only depends on $X(x)$ and $Y(x)$ and f. In other words we WTS that if $X(x) = \tilde{X}(x)$ and $Y(x) = \tilde{Y}(x)$

$((XY)(f))(x) = ((\tilde{X}\tilde{Y})(f))(x)$

To show this equality first we need to understand that

$$df(p)(X(p)) = (X(f))(p) \qquad \forall X \in \mathfrak{X}(U) \quad \forall p \in U$$

why? Because let $X = \sum a^i \frac{\partial}{\partial x_i}$ $\quad (X(f))(p) = \sum a^i(p) \frac{\partial f}{\partial x_i}(p)$

. The map $df_p : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$$q \longmapsto \left(\frac{\partial f}{\partial x_1}(p)\, dx_1 + \cdots + \frac{\partial f}{\partial x_n}(p)\, dx_n\right)(q) = \sum \frac{\partial f}{\partial x_i}(p)\, dx_i(q)$$

$q = X(p)$

takes the point "$(a^1(p), \ldots, a^n(p))$ to $\frac{\partial f}{\partial x_1}(p)\, a^1(p) + \cdots + \frac{\partial f}{\partial x_n} a^n(p)$

Thus we have $df(p)(X(p)) = (X(f))(p)$. /

now we have $(X(\underbrace{Y(f)}_{g \in C^\infty(U)}))(x) = dg(x)(X(x)) = dg(x)(\tilde{X}(x)) = (\tilde{X}(g))(x)$

$= (\tilde{X}(Y(f)))(x) = (Y(\underbrace{\tilde{X}(f)}_{h}))(x) = dh(x)(Y(x)) = dh(x)(\tilde{Y}(x)) =$

bcz x is critical

$(\tilde{Y}(h))(x) = (\tilde{Y}(\tilde{X}(f)))(x) = (\tilde{X}(\tilde{Y}(f)))(x)$. /

Let $v = X(x)$ and $w = Y(x)$, define the map $Hf(x) : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$

(x is still the critical pt)

$(v, w) \longmapsto X(Y(f))(x)$

we want to show that this is a symmetric 2-0 tensor. i.e.

This is ① well defined.

② bilinear

③ Symmetric.

① This is well-defined because we have shown that for $(v, w) = (\tilde{v}, \tilde{w})$

$Hf(x)(\tilde{v}, \tilde{w}) = Hf(x)(v, w)$

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② this is linear in v bcz $Hf(x)(\lambda v_1 + v_2, w) = \lambda Hf(x)(v_1, w) + Hf(x)(v_2, w)$

where $\lambda \in \mathbb{R}$, $v_1 = X_1(x)$ $v_2 = X_2(x)$ $w = Y(x)$ $X_1, X_2, Y \in \mathfrak{X}(U)$.

$\rightsquigarrow Hf(x)(\lambda v_1 + v_2, w) = Hf(x)(\lambda X_1(x) + X_2(x), Y(x)) = Hf(x)((\lambda X_1)(x) + X_2(x), Y(x))$

(Check the vector space structure of $\mathfrak{X}(U)$ ← I wrote before.)

$= Hf(x)((\lambda X_1 + X_2)(x), Y(x)) = ((\lambda X_1 + X_2) \circ Y(f))(x)$

by def. of $Hf(x)$ composition

$= ((\lambda X_1) Y(f))(x) + (X_2(Y(f)))(x) = \lambda (X_1(Y(f)))(x) + (X_2(Y(f)))(x)$

$= \lambda Hf(x)(X_1(x), Y(x)) + Hf(x)(X_2(x), Y(x))$

$= \lambda Hf(x)(v_1, w) + Hf(x)(v_2, w)$

this is linear in w because it is symmetric :

③ WTS $Hf(x)$ is symmetric, i.e. $Hf(x)(v, w) = Hf(x)(w, v)$

$v = X(x)$ $w = Y(x)$

we have shown before bcz x is critical pt.

$Hf(x)(v, w) = X(Y(f))(x) = Y(X(f))(x) = Hf(x)(w, v)$.

$\boxplus$

Ⓑ we show that if x is not a critical pt. then $Hf(x)$ is not well defined by giving a counter example :

let $U = \mathbb{R}^2$, $f : \mathbb{R}^2 \to \mathbb{R}$, $(x_1, x_2) \mapsto x_1$, $Y = a^1 \frac{\partial}{\partial x_i} + a^2 \frac{\partial}{\partial x_2} \in \mathfrak{X}(U)$

and $\tilde{Y} = b^1 \frac{\partial}{\partial x_1} + b^2 \frac{\partial}{\partial x_2} \in \mathfrak{X}(U)$ with $\begin{cases} a^1 : \mathbb{R}^2 \to \mathbb{R} \\ (x_1, x_2) \mapsto x_1 \\ a^2 : \mathbb{R}^2 \to \mathbb{R} \\ (x_1, x_2) \mapsto 2x_2 \end{cases}$

and $\begin{cases} b^1 : \mathbb{R}^2 \to \mathbb{R} \\ (x_1, x_2) \mapsto 2x_1 - x_2 \\ b^2 : \mathbb{R}^2 \to \mathbb{R} \\ (x_1, x_2) \mapsto x_2 + x_1 \end{cases}$ and let $p = (1, 1) \in \mathbb{R}^2$

observe that $\begin{cases} a^1(p) = b^1(p) \\ a^2(p) = b^2(p) \end{cases} \Rightarrow Y(p) = \tilde{Y}(p)$

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And let $X = \frac{\partial}{\partial x_1}$.   see it as a function $(x_1, x_2) \to x_1$

we have $(X(Y(f)))(p) = \left(X\left(a^1 \frac{\partial f}{\partial x_1} + a^2 \frac{\partial f}{\partial x_2}\right)\right)(p) = (X(x_1))(p)$

$= \left(1 \frac{\partial x_1}{\partial x_1}\right)(p) = (1)(p) = \boxed{1}$

the constant fuction $(x_1, x_2) \mapsto 1$

$(X(\tilde{Y}(f)))(p) = \left(X\left(b^1 \frac{\partial f}{\partial x_1} + b^2 \frac{\partial f}{\partial x_2}\right)\right)(p) = (X(2x_1 - x_2))(p)$

$= \left(\frac{\partial(2x_1 - x_2)}{\partial x_1}\right)(p) = 2(p) = \boxed{2}$

the constant function 2.

so they are not equal and the map is not well-defined. /

Problem 4.c was not correct. The new version of that with solution is also uploaded. You do not lose point for this problem.