Series 9

1. Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space and $C \subset \mathcal{H}$ a closed and convex set, i.e.

$$v, w \in C \;\Rightarrow\; tv + (1-t)w \in C \text{ f.a. } 0 \le t \le 1\,.$$

Show: For all $x \in \mathcal{H} \setminus C$ there exists a unique $y \in C$ such that

$$\|x - y\| \;=\; \inf\{\, \|x - v\| \mid v \in C\,\}\;.$$

Hint: Use the parallelogramm law from the inner product:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2, \quad \text{f.a. } x, y \in \mathcal{H}\,.$$

2. **a)** Show that every closed subspace $V \subset \mathcal{H}$ of a Hilbert space $\mathcal{H}$ possesses an orthogonal closed complement

$$\mathcal{H} = V \oplus V^\perp, \quad V^\perp \;=\; \{\, w \in \mathcal{H} \mid \langle w, v\rangle = 0 \text{ f.a. } v \in V\,\}\;,$$

i.e. $V \cap V^\perp = \{0\}$ and $V + V^\perp = \mathcal{H}$. *2 points*

b) Consider $T\colon \mathcal{H} \longrightarrow \mathcal{H}^*$, $T(v) = \langle v, \cdot\rangle$. Show that $\|T(v)\|_{Op} = \|v\|$ f.a. $v \in \mathcal{H}$ and that T is an isomorphism. *2 points*

3. Consider the polynomials $f_n(x) = x^n$ as continuous functions on the intervall $[0,1]$ and let

$$D \;:=\; \bigcup_{n\in\mathbb{N}} \operatorname{span}\{f_0, \dots, f_n\} \;\subset C^0([0,1])\,.$$

The Theorem of Stone-Weierstrass yields that D is dense in $C^0([0,1])$ w.r.t. the sup-norm $\|\cdot\|_\infty$.

a) Deduce that D is also dense in the Hilbert space $\mathcal{H} = L^2([0,1])$ and show that for all $f \in C^0([0,1])$ we have

$$\int_0^1 f(x)x^n\,dx \;=\; 0 \quad \text{f.a. } n \ge 0 \quad \Rightarrow \quad f \equiv 0\,.$$

2 points

b) Apply the Gram-Schmidt algorithm to the sequence $(f_n)_{n\ge 0}$ in order to obtain the Hilbert basis $(e_n)_{n\ge 0}$ and compute e_0, e_1 and e_2. *2 points*

4. Let $f \in L^1(\mathbb{R}) \cap C^1(\mathbb{R})$ with $f' \in L^1(\mathbb{R})$. Suppose that in addition we have

$$(*) \qquad \lim_{|x|\to\infty} \big|f(x)\big| \;=\; 0\,.$$

a) Show that

$$\widehat{f'}(k) \;=\; 2\pi i k \hat{f}(k) \quad \text{f.a. } k \in \mathbb{R}\,.$$

2 points

b) Is this consequence also true without the assumption $(*)$? *2 points*

Hand-In: Practice Session Thursday Jan 16, 2020