Solutions to problem Sheet 8

(1) $\mathcal{T}$ is a collection of open subsets of $\mathbb{R}^N$ and $s>0$.

$$\tau_s : \mathcal{T} \longrightarrow [0,\infty]$$
$$U \longmapsto (\operatorname{diam}(U))^s \qquad \operatorname{diam} U = \sup_{x,y\in U} \underbrace{|x-y|}_{d(x,y)}$$

Let $\mathcal{T}_\xi = \{T\in\mathcal{T} \mid \operatorname{diam} T<\xi\}$ and consider the out. meas:

$$\mu^*_{\xi,s}(A) := \inf\left\{\sum_{n=1}^{\infty} \tau_s(U_n) \,\middle|\, U_n\in\mathcal{T}_\xi,\, A\subset \bigcup_{n=1}^{\infty} U_n\right\}$$

(a) show that $\underset{E\subset\mathbb{R}^n}{0<\xi'<\xi} \Rightarrow \mu^*_{\xi',s}(E) \geq \mu^*_{\xi,s}(E)$

$$\xi'<\xi \Rightarrow \mathcal{T}_{\xi'} \subset \mathcal{T}_\xi \Rightarrow \left\{U_n\in\mathcal{T}_{\xi'},\, E\subset\bigcup_{n=1}^{\infty} U_n\right\} \subset \left\{U_n\in\mathcal{T}_\xi,\, E\subset\bigcup_{n=1}^{\infty} U_n\right\}$$

$$\Rightarrow \left\{\sum_{n=1}^{\infty}\tau_s(U_n) \,\middle|\, U_n\in\mathcal{T}_{\xi'},\, E\subset\bigcup_{n=1}^{\infty}U_n\right\} \subset \left\{\sum_{n=1}^{\infty}\tau_s(U_n) \,\middle|\, U_n\in\mathcal{T}_\xi,\, E\subset\bigcup_{n=1}^{\infty}U_n\right\}$$

note $B\subset C \Rightarrow \inf B \geq \inf C$
two sets

$\Rightarrow \inf$ 〃 $\geq \inf$ 〃

$\Rightarrow \mu^*_{\xi',s}(E) \geq \mu^*_{\xi,s}(E)$

WTS $\mu^*_{(s)}(E) = \lim_{\xi\to 0}\mu^*_{\xi,s}(E)$ is well-defined $\forall E\subset\mathbb{R}^N$.

let $f(\xi) = \mu^*_{\xi,s}(E)$ $\forall E\subset\mathbb{R}^n$ $\quad f:(0,+\infty)\to\mathbb{R}_{\geq 0}$

~~decreasing~~ ξ increases (not decreases) $f(\xi)$ that is a real valued function in ξ therefore ~~$f(\xi)$ is a real valued~~ f is a real valued monotonic and function $\lim_{\xi\to 0} f(\xi)$ exists though can be $+\infty$.

(b) Given that $\mu^*_{\xi,S}$ is an outer meas. we WTS that $\mu^*_{(S)}$ is an outer meas:

we are using it.

i.e. WTS:

(1) $\mu^*_{(S)}(\emptyset) = 0$

(2) $\mu^*_{(S)}(A) \leq \mu^*_{(S)}(B)$ if $A \subseteq B$ $\forall A, B \subset \mathbb{R}^N$

(3) $\mu^*_{(S)}\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*_{(S)}(A_n)$ if $A_i \in \mathbb{R}^N$ $\forall i \in \{1,2,\dots\}$

(1) $\mu^*_{(S)}(\emptyset) = \lim_{\xi \to 0} \mu^*_{\xi,S}(\emptyset) = \lim_{\xi \to 0} 0 = 0$ ✓

(2) $A \subseteq B \Rightarrow \mu^*_{\xi,S}(A) \leq \mu^*_{\xi,S}(B) \Rightarrow \lim_{\xi \to 0} \mu^*_{\xi,S}(A) \leq \lim_{\xi \to 0} \mu^*_{\xi,S}(B)$

$\Rightarrow \mu^*_{(S)}(A) \leq \mu^*_{(S)}(B)$ ✓

(3) $\mu^*_{\xi,S}\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu^*_{\xi,S}(A_i) \Rightarrow$

from (a)

$\lim_{\xi \to 0} \mu^*_{\xi,S}\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \lim_{\xi \to 0} \sum_{i=1}^{\infty} \mu^*_{\xi,S}(A_i) \leq \lim_{\xi \to 0} \sum_{i=1}^{\infty} \underbrace{\lim_{\xi \to 0} \mu^*_{\xi,S}(A_i)}$

$\parallel$

$\mu^*_{(S)}\left(\bigcup_{i=1}^{\infty} A_i\right)$

$= \lim_{\xi \to 0} \sum_{i=1}^{\infty} \mu^*_{(S)}(A_i)$

$= \sum_{i=1}^{\infty} \mu^*_{(S)}(A_i)$ ✓

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ⓒ WTS $\mu^*_{(S)}(T) = \begin{cases} 0 & \text{if } S > N \\ \infty & \text{if } S < N \end{cases}$ where $\underset{\text{open}}{T \subset \mathbb{R}^N}$

idea of the proof.

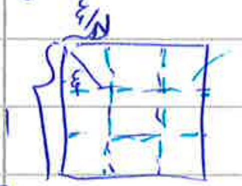
first we show that for an open ball (here is open hypercube) Q we have

$$\mu^*_{(S)}(Q) = \begin{cases} 0 & \text{if } S > N \\ \infty & \text{" } S < N \end{cases}.$$

later we show that this equality also holds for $\mathbb{R}^N$.

Then since any open subset T of $\mathbb{R}^N$ contains an open ball and is contained in $\mathbb{R}^N$, the result follows from (b2)

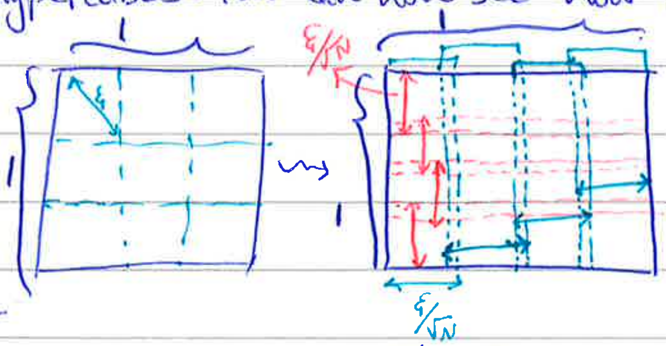
WLOG, let Q be a hypercube of side length one, i.e. a unit hypercube you can partition Q into $\underset{\text{open}}{S}$ hypercubes of $\underset{=\text{diagonal}}{\text{diameter}} = \xi$



here $S = \left(\frac{1}{\frac{\xi}{\sqrt{N}}}\right)^N = \left(\frac{\sqrt{N}}{\xi}\right)^N$. but there is a problem

Note that in this picture $\boxed{N=2}$

Since we want our covering to be open and cover the whole cube but this S hypercubes of diagonal $= \xi$ are not covering the boundaries. we can solve this problem by taking $S' = \left(\frac{1}{\frac{\xi}{\sqrt{N}}} + 1\right)^N$ of such hypercubes. You can now see how this can cover everything:



now you have

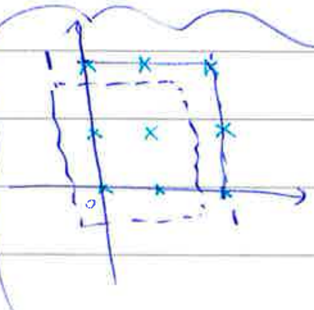
$$\mu^*_{\xi,S}(Q) \le \left(\frac{1}{\frac{\xi}{\sqrt{N}}} + 1\right)^N \cdot \xi^S$$

$$\Rightarrow \mu^*_{\xi,S}(Q) \le \frac{(\sqrt{N}+\xi)^N}{\xi^N} \cdot \xi^S = (\sqrt{N}+\xi)^N \cdot \xi^{S-N}$$ let $\xi \to 0$ for $S > N$

$\Rightarrow \mu^*_{(S)}(Q) \le 0 \Rightarrow$ for $S > N$, $\boxed{\mu^*_{(S)}(Q) = 0}$

Now we use a method including "counting points" to prove that for $s<N$, $\mu^*_{(s)}(Q)=\infty$.

let $k=\frac{1}{n}$, $n\in\mathbb{N}$ fixed, define $k\mathbb{Z}^N=\{kz : z\in\mathbb{Z}^N\}$. The number of $k\mathbb{Z}^N$ points inside a unit hypercube is at most $(\frac{1}{k})^N$.



$k=\frac{1}{2}$, $N=2$

x → are points of $k\mathbb{Z}^2$

you see that Q can at most have $\left(\frac{1}{\frac{1}{2}}\right)^2=4$ of those pts.

Now let $U\underset{\text{open}}{\subset} Q$ be an open subset of Q with diam $\leq \epsilon$.

You can find a hypercube Q_ϵ with the side length $=2\,\mathrm{diam}(U)$

now with the same argument that we had for the unit hypercube we also know that the number of $k\mathbb{Z}^N$ points in Q_ϵ is at most $\left(\frac{2\,\mathrm{diam}\,U}{k}\right)^N$

therefore $\boxed{\#\, k\mathbb{Z}^N \text{ pts in } U \leq \left(\frac{2\,\mathrm{diam}\,U}{k}\right)^N.}$

for any cover with such subsets $U\subset Q$ we have

$$\left(\frac{1}{k}\right)^N \leq \sum_U \left(\frac{2\,\mathrm{diam}\,U}{k}\right)^N = \left(\frac{2}{k}\right)^N \sum_U (\mathrm{diam}\,U)^N$$

$$\Rightarrow \underbrace{\sum_U (\mathrm{diam}\,U)^N}_{} \geq \left(\frac{1}{2}\right)^N$$

$$\overbrace{\sum_U (\mathrm{diam}\,U)^{N-s}(\mathrm{diam}\,U)^s}^{\parallel} \leq \epsilon^{N-s}\sum_U (\mathrm{diam}\,U)^s$$

$$\Rightarrow \sum_U (\text{diam } U)^s \geq \frac{1}{2^N} \cdot \frac{1}{\varepsilon^{N-s}}$$

This inequality holds for any cover of Q where $\text{diam } U < \varepsilon$.

letting $\varepsilon \to 0$, for $s < N$ we have $\sum_U (\text{diam } U)^s \geq \infty$

$$\boxed{\lim_{\varepsilon \to 0} \mu^*_{\varepsilon, s}(Q) = \infty \Rightarrow \mu^*_{(s)}(Q) = \infty \text{ for } s < N.}$$

Next we want to show the equalities for $\mathbb{R}^N$.

let $\boxed{s < N}$ since $Q \subset \mathbb{R}^N$, from (b)(2) $\mu^*_{(s)}(\mathbb{R}^N) \geq \mu^*_{(s)}(Q) = \infty$

$$\boxed{\Rightarrow \mu^*_{(s)}(\mathbb{R}^N) = \infty}$$

Now let $\boxed{s > N}$; $\mathbb{R}^N$ can be covered by countably many unit hypercubes Q_i.

by (b)(3) we have:

$$\boxed{\mu^*_{(s)}(\mathbb{R}^N) = \mu^*_{(s)}\left(\bigcup_{i=1}^{\infty} Q_i\right) \leq \sum_{i=1}^{\infty} \mu^*_{(s)}(Q_i) = 0}$$

(d) Let $E \subset \mathbb{R}^n$ with $\mu^*_{(s)}(E) < \infty$ and $t > s$, WTS $\mu^*_{(t)}(E) = 0$

let $\varepsilon > 0$. we have $\mu^*_{\varepsilon, s}(E) \underset{\text{by (a)}}{\leq} \mu^*_{(s)}(E) < \infty$.

Let $\{U_i\}$ be countably many open sets s.t. $\text{diam } U_i < \varepsilon$,

$A \subset \bigcup_{i=1}^{\infty} U_i$ and $\sum (\text{diam } U_i)^s \leq \mu^*_{\varepsilon, s}(E) + 1 \leq \mu^*_{(s)}(E) + 1$,

Thus this gives us,

$$\mu^*_{\varepsilon, t}(E) \leq \sum (\text{diam } U_i)^t = \sum (\text{diam } U_i)^{t-s} (\text{diam } U_i)^s$$

$$\leq \varepsilon^{t-s} \sum (\text{diam } U_i)^s$$

$$\leq \varepsilon^{t-s} \underbrace{(\mu^*_{(s)}(E) + 1)}_{< \infty} \Bigg\} \xrightarrow{\varepsilon \to 0} 0$$

$$\Rightarrow \mu^*_{(t)}(E) = 0$$

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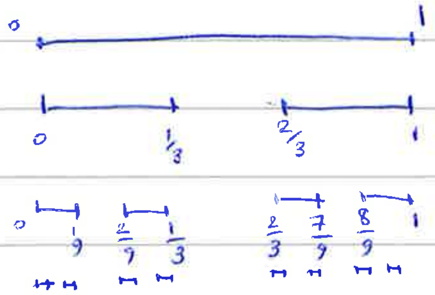
2 - consider the cantor-set $C = \left\{ \sum_{n=1}^{\infty} \frac{a_n}{3^n} \;\middle|\; a_n \in \{0,2\} \text{ f.a. } n \in \mathbb{N} \right\}$

first let's see that why this set is the same as the geometrical object that we know as :



what we get after doing this ← process for infinite number of times is the Set C. why?

let's see some of the points that are in C:

* taking all a_n to be 0 gives us $0 \Rightarrow 0 \in C$
* " " " " " 2 " " $\sum_{n=1}^{\infty} \frac{2}{3^n} = 2 \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n = 2 \cdot \frac{\frac{1}{3}}{1-\frac{1}{3}} = \boxed{1} \Rightarrow 1 \in C$
* take $a_1 = 0$ and $a_n = 2$ $\forall n \geq 2$: $\sum_{n=2}^{\infty} \frac{2}{3^n} = 2 \sum_{n=2}^{\infty} \left(\frac{1}{3}\right)^n = 2 \cdot \frac{\frac{1}{9}}{1-\frac{1}{3}} = \boxed{\frac{1}{3}} \in C$
* take $a_1 = 2$ and $a_n = 0$ $\forall n \geq 2$: $\sum_{n=1}^{\infty} \frac{a_n}{3^n} = \boxed{\frac{2}{3}} \in C$

you can simply see that continuing the geometrical process above also will keep these 4 points until infinity... Now you have some feelings that why they are actually the same... to see it properly :

Any element in C is of the form $x = \frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \cdots$ where $a_n \in \{0,2\}$ so

writting this number in the base of $\underline{3}$ gives you $x = 0.a_1 a_2 a_3 \cdots$ where $a_n \in \{0,2\}$ let's see the two possibilities : $\begin{cases} a_1 = 0 \\ \text{or} \\ a_1 = 2 \end{cases}$

if $a_1 = 0$ x varies between $\underbrace{0.00\cdots}_{} \overset{0}{\shortparallel} \leq x \leq 0.022\cdots = \sum_{n=2}^{\infty} \frac{2}{3^n} = \frac{1}{3}$

if $a_1 = 2$ " " " $0.200\cdots \leq x \leq 0.22\cdots = \sum_{n=1}^{\infty} \frac{2}{3^n} = 1$

$\frac{2}{3}$ "

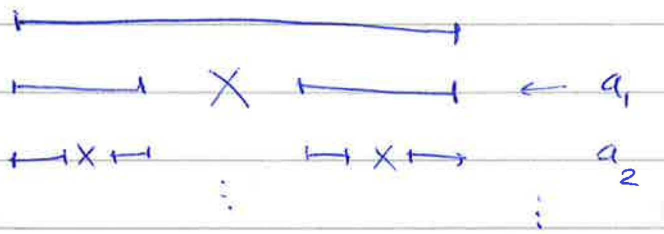
Thus $x \begin{cases} \nearrow a_1 = 0 \to 0 \le x \le \frac{1}{3} \\ \searrow a_1 = 2 \to \frac{2}{3} \le x \le 1 \end{cases}$ $\to$ you see that x is never in $(\frac{1}{3}, \frac{2}{3})$

now you go to the next digit a_2:

$$x \begin{cases} \nearrow a_1 = 0 \begin{cases} \nearrow a_2 = 0 \longrightarrow 0 \le x \le \frac{1}{9} \\ \searrow a_2 = 2 \longrightarrow \frac{2}{9} \le x \le \frac{1}{3} \end{cases} \\ \searrow a_1 = 2 \begin{cases} \nearrow a_2 = 0 \longrightarrow \frac{6}{9} \le x \le \frac{7}{9} \\ \searrow a_2 = 0 \longrightarrow \frac{8}{9} \le x \le 1 \end{cases} \end{cases}$$

So you can see that x can never be in

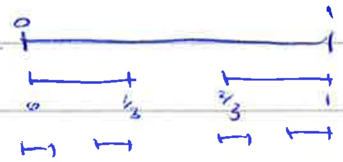
$$(\tfrac{1}{9}, \tfrac{2}{9}) \cup \underbrace{(\tfrac{1}{3}, \tfrac{2}{3})}_{\text{this you knew before}} \cup (\tfrac{7}{9}, \tfrac{8}{9})$$



doing the same process on all a_n, $n \to \infty$ gives exactly the geometrical picture we have.

(a) WTS that C is the intersection of compact subsets $C_1 \supset C_2 \supset \dots$.

you can alread see from the picture that what you can take as your C_i:



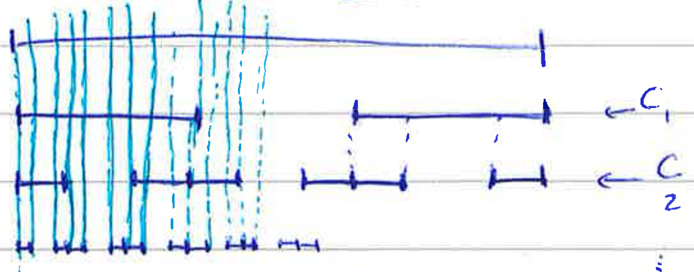
$\to C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$

$\to C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$

each C_i is closed and bounded, thus compact.

and $\bigcap_{i=1}^{\infty} C_i = C$.

you can find a formula for "similar" C_i's:



intersecting C_i's obviously gives the same result as before which is the Cantor set.

$$C_i = \bigcup_{k=0}^{3^{i-1}-1} \left(\left[\frac{3k+0}{3^i}, \frac{3k+1}{3^i} \right] \cup \left[\frac{3k+2}{3^i}, \frac{3k+3}{3^i} \right] \right)$$

C_i is a finite union of closed subset $\Rightarrow C_i$ is closed
$\forall_i \quad C_i \subset [0,1] \Rightarrow C_i$ is bdd
$\Rightarrow C_i$ is compact $\forall_i$

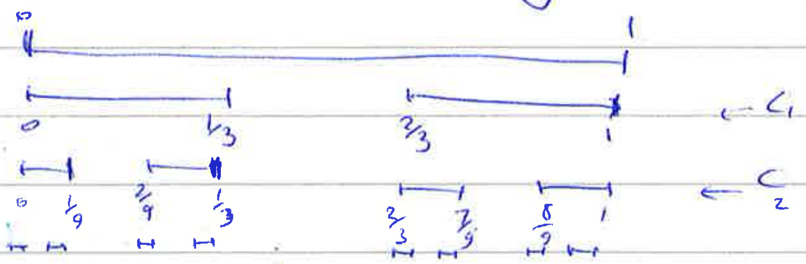
and $C = \bigcap_{i=1}^{\infty} C_i$. /

(b) Let λ' be the Lebesgue measure and C' be the set $[0,1] \setminus C \subseteq [0,1]$

$$\underbrace{\lambda'([0,1])}_{1} = \lambda'(C \cup C') = \lambda'(C) + \lambda'(C') \quad (*)$$

we only need to compute $\lambda'(C')$.

C' which is the compliment of the cantor set C in $[0,1]$ is the countable union of all the intervals that we are removing at each step.



The length of the removed interval at first step is $\frac{1}{3}$
" " " " " " " second " " $2 \times \frac{1}{9}$
" " " " " " " third " " $2^2 \times \frac{1}{3^3}$
⋮

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Therefore $\lambda'(C') = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = \frac{1}{2} \sum \left(\frac{2}{3}\right)^n = \frac{1}{2} \frac{\frac{2}{3}}{1-\frac{2}{3}} = 1$]

now from $(*)$ we have $\lambda'(C) = 0$.

(c) WTS $\mu^*_{(s)}(C) \leq 1$ for $s = \frac{\log 2}{\log 3}$

let $E_1 = \left(0 - \frac{\delta}{2}, \frac{1}{3} + \frac{\delta}{2}\right) \cup \left(\frac{2}{3} - \frac{\delta}{2}, 1 + \frac{\delta}{2}\right)$

$$E_2 = \left(0 - \frac{\delta}{2}, \frac{1}{9} + \frac{\delta}{2}\right) \cup \left(\frac{2}{9} - \frac{\delta}{2}, \frac{1}{3} + \frac{\delta}{2}\right) \cup \left(\frac{2}{3} - \frac{\delta}{2}, \frac{7}{9} + \frac{\delta}{2}\right) \cup \left(\frac{8}{9} - \frac{\delta}{2}, 1 + \frac{\delta}{2}\right)$$

$\vdots$

You can simply see that the cantor set C is contained in all E_k for $\delta > 0$. Let $\varepsilon = 3^{-k} + 2\delta$. Each $E_i = \{U_{ij}\}_{j=1}^{2^i}$ is a covering for C where diam $U_{ij} \leq \varepsilon$. for all k we have

$$\mu^*_{\varepsilon,s}(C) \leq \mu^*_{\varepsilon,s}(E_k) \leq \sum_{j=1}^{2^k} \mu^*_{\varepsilon,s}(U_{kj}) \leq \sum_{j=1}^{2^k} (3^{-k} + 2\delta)^s$$

U_{kj} is a covering of itself

but since $s = \frac{\log 2}{\log 3} = \log_3 2$ we have $\boxed{3^s = 2}$

therefore $\mu^*_{\varepsilon,s}(C) \leq 2^k \cdot (3^{-k} + 2\delta)^s = 2^k \cdot \left(3^{-ks} + a_1 3^{-ks-1}(2\delta) + a_2 \ldots + (2\delta)^s\right)$

$$= \underbrace{2^k \cdot (3^s)^{-k}}_{1}$$

$$+ \; 2^k \cdot \left(a_1 3^{-ks-1}(2\delta) + a_2 3^{-ks-2}(2\delta)^2 + \cdots + (2\delta)^s\right)$$

let $\delta = \frac{1}{2^k}$, $k \to \infty$ or equivalently $\varepsilon \to 0$ implies

$$\boxed{\mu^*_s(C) \leq 1}.$$

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(d) WTS $\mu^*_{(s)}(C) \ge \frac{1}{2} = 3^{-s}$ where $s = \frac{\log 2}{\log 3} = \log_3 2$.

we ~~can~~ show that for every covering $\{U_i\}_{i=1}^{\infty}$ of C we have $\sum_{i=1}^{\infty} |U_i|^s \ge 3^{-s}$. we can assume that U_i's are intervals (if not then the sum is even bigger.) $\left(|U_i| := \operatorname{diam} U_i \right)$

[using the compactness of C you can assume this is a finite covering]

for each U_i let k be the integer s.t.

$$(**) \qquad 3^{-(k+1)} \le |U_i| < 3^{-k}$$

U_i intersects at most 1 level k-interval. If $j \ge k$ then U_i intersects at most $2^{j-k} = 2^j \, 2^{-k} = 2^j \left(3^{-sk}\right)$

from $(**)$ $\left(3^{-(k+1)}\right)^s \le |U_i|^s \Rightarrow 3^{-ks-s} \le |U_i|^s$

$\Rightarrow 3^{-sk} \le 3^s |U_i|^s$

$\le 2^j \cdot 3^s \cdot |U_i|^s$ intervals of level k.

let j be large enough s.t. $3^{-(j+1)} \le |U_i| \quad \forall i$.

we know that $\{U_i\}$ is intersecting 2^j level j intervals. ($\{U_i\}$ is a covering of C!)

counting intervals:

$$2^j \le \sum_{i=1}^{n} 2^j \cdot 3^s \cdot |U_i|^s \Rightarrow \sum_{i=1}^{n} |U_i|^s \ge 3^{-s} = \frac{1}{2} \checkmark$$

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3 _ let $f:\mathbb{R}\to\mathbb{R}$ be the function $f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 1 \end{cases}$

WTS that f is not Lebesgue-integrable.

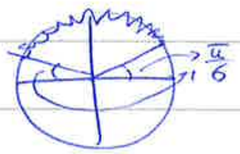
By definition f is ~~int~~ Lebesgue integrable $\Longleftrightarrow$ f^+ and f^- are Lebesgue-integrable.

Remember: $f^+(x) = \max\{f(x), 0\}$

since f^+ is positive and is continuous and therefore

Riemann-integrable we have: $\int_{\mathbb{R}} f^+(x)\,d\lambda = \int_{\mathbb{R}} f^+(x)\,dx$

let $A = \bigcup_{k=1}^{\infty} \left[\left(\frac{1}{6}+2k\right)\pi, \left(\frac{5}{6}+2k\right)\pi\right]$



$\forall x \in A$ we have $\sin x \geq \frac{1}{2} \Rightarrow f^+(x) \geq \frac{1}{2x}$ $\forall x \in A$.

$$\int_{\mathbb{R}} f^+(x)\,dx \geq \int_A f^+(x)\,dx = \sum_{k=1}^{\infty} \int_{(\frac{1}{6}+2k)\pi}^{(5/6+2k)\pi} f^+(x)\,dx \geq \sum_{k=1}^{\infty} \int_{(\frac{1}{6}+2k)\pi}^{(\frac{5}{6}+2k)\pi} \left(\frac{1}{2x}\right)dx$$

$$\geq \frac{1}{2} \sum_{k=1}^{\infty} \int_{(\frac{1}{6}+2k)\pi}^{(5/6+2k)\pi} \frac{1}{3k\pi} = \frac{1}{2} \cdot \sum_{k=1}^{\infty} \frac{4}{6}\cancel{\pi} \cdot \frac{1}{3k\cancel{\pi}}$$

$\hookrightarrow$ $3k\pi > (\frac{5}{3}+2k)\pi$ for $k \geq 1$

$$= \frac{1}{9} \sum_{k=1}^{\infty} \frac{1}{k} > \infty$$

$\hookrightarrow$ we know

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4.(a). $f_n : \mathbb{R} \to \mathbb{R}$ $f_n = \frac{1}{(1+x^2)^n}$ we want to see whether

$\lim_{n\to\infty} \int_{\mathbb{R}} f_n \, d\lambda'$ exists and wether $\lim_{n\to\infty}\int_{\mathbb{R}} f_n \, d\lambda' = \int_{\mathbb{R}} \lim_{n\to\infty} f_n \, d\lambda'$.

we use dominated convergence theorem:

suppose $f_n \to f$ pointwise μ-a.e. and suppose there exists $G : X \to [0,\infty]$, such that G is μ integrable and $|f_n(x)| \leq G(x)$ μ-a.e. $\forall n \in \mathbb{N}$.
Then
$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu$$

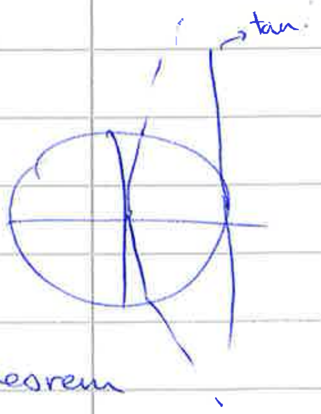
we have $\lim_{n\to\infty} \frac{1}{(1+x^2)^n} = \begin{cases} 1 & x=0 \\ 0 & x \neq 0 \end{cases}$, therefore $f_n \to f$ pointwise

where $f = \mathbb{1}_{\{0\}}$. This is Lebesgue integrable,

$A \subseteq \mathbb{R}$ $\int \mathbb{1}_A \, d\mu = \mu(A)$
$\mathbb{1}_A = \begin{cases} 0 & x \notin A \\ 1 & x \in A \end{cases}$

Thus $\int_{\mathbb{R}} f \, d\lambda$ exists and is $\boxed{\lambda(\{0\}) = 0}$. So we have $\boxed{\int \lim_{n\to\infty} f_n \, d\lambda = 0}$

Also $\forall n \in \mathbb{N}$ and $x \in \mathbb{R}$ we have $|f_n| = \left| \frac{1}{(1+x^2)^n} \right| \leq \frac{1}{1+x^2} = f_1(x)$



Since f_1 is continuous and nonnegative $\int_{\mathbb{R}} f_1 \, d\lambda = \int_{-\infty}^{\infty} f_1 \, dx$

$= \int_{-\infty}^{\infty} \frac{1}{1+x^2} \, dx = \lim_{R\to\infty} \tan^{-1} \Big]_{-R}^{R} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi < \infty$

$\Rightarrow f_1$ is Lebesgue integrable. Using dominated converges theorem we see that exists and

we have $\boxed{\lim_{n\to\infty} \int f_n \, d\lambda = \int f \, d\mu = 0}$ /

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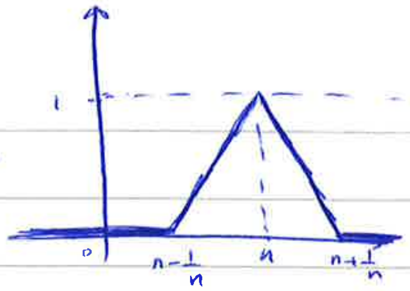
Now consider $f_n(x) = \begin{cases} 0, & x \leq n - \frac{1}{n}, \\ nx + 1 - n^2, & n - \frac{1}{n} \leq x \leq n, \\ 1 + n^2 - nx, & n \leq x \leq n + \frac{1}{n}, \\ 0, & x \geq n + \frac{1}{n}. \end{cases}$



Let's first compute $\int_{\mathbb{R}} f_n \, d\lambda$:

$$\int_{\mathbb{R}} f_n \, d\lambda = \underbrace{\cancel{\int_{(-\infty,0)} f_n \, d\lambda}}_{0} + \int_{[0,+\infty)} f_n \, d\lambda = \int_{[0,+\infty)} f_n \, d\lambda$$

$$= \int_0^\infty f_n(x) \, dx = \int_{n-\frac{1}{n}}^{n} (nx + 1 - n^2) \, dx + \int_n^{n+\frac{1}{n}} (1 + n^2 - nx) \, dx$$

$$\left[n \frac{x^2}{2} + (1-n^2)x \right]_{n-\frac{1}{n}}^{n} + \left[(1+n^2)x - n\frac{x^2}{2} \right]_n^{n+\frac{1}{n}}$$

$$= \frac{1}{n} \implies \lim_{n \to \infty} \int_0^\infty f_n \, d\lambda = \lim_{n \to \infty} \frac{1}{n} = \boxed{0}$$

On the other hand we also have $\lim f_n = 0 \implies \boxed{\int_0^\infty \lim f_n \, d\lambda = 0}$

for $x \leq n - \frac{1}{n}$ $f_n(x) = 0$ let $n \to \infty$ $\Rightarrow$ for $x < \infty$ $f_n(x) = 0$ $\}$ $n \to \infty$

So both limits exist and are equal.

4.b) WTS for $(\mathbb{N}, P(\mathbb{N}), \mu_{count})$

$$1 \leq r < s \implies L^r(\mathbb{N}, \mu_{count}) \subset L^s(\mathbb{N}, \mu_{count}.)$$

Let $f \in L^r(\mathbb{N}, \mu_{count})$ i.e. f is μ_{count} - measurable and

$\int_{\mathbb{N}} |f|^r \, d\mu_{count} < \infty$. Note that $\int_{\mathbb{N}} |f|^r \, d\mu_{count} = \sum_{n=1}^\infty |f(n)|^r$.

therefore $\sum_{n=1}^\infty |f(n)|^r < \infty \implies \lim_{n \to \infty} f(n) = 0 \implies \begin{cases} \text{for only finitely many } n: \\ |f(n)| \geq 1. \end{cases}$

$$\Rightarrow \int_{\mathbb{N}} |f|^s d\mu_{count} = \sum_{n=1}^{\infty} |f(n)|^s = \sum_{\substack{1\le n<\infty \\ \text{with } |f(n)|<1}} |f(n)|^s + \boxed{\sum_{\substack{1\le n<\infty \\ \text{with } |f(n)|\ge 1}} |f(n)|^s}$$

↳ this is a bdd number $M < \infty$

$r<s$

$$= \sum_{\substack{1\le n<\infty \\ \text{with } |f(n)|<1}} |f(n)|^s + M \le \sum_{\substack{1\le n<\infty \\ \text{with } |f(n)|<1}} |f(n)|^r + M$$

remember $(\frac{1}{10})^3 < (\frac{1}{10})^2$!

$$\le \sum_{n=1}^{\infty} |f(n)|^r = \int_{\mathbb{N}} |f|^r d\mu_{count}.$$

therefore $\int_{\mathbb{N}} |f|^r d\mu_{count} < \infty \Rightarrow \int_{\mathbb{N}} |f|^s d\mu_{count} < \infty$. /

Now WTS for $([0,1], \mathcal{B}([0,1]), \lambda')$:

$$1 \le r < s \Rightarrow L^r([0,1], \lambda') \supset L^s([0,1], \lambda')$$

Let $f \in L^s([0,1], \lambda')$ i.e. f is λ'-measurable and

$\int_{[0,1]} |f|^s d\lambda' < \infty \Rightarrow$

$$\int_{[0,1]} |f|^r d\lambda' = \int_{\substack{0 \\ |f|\le 1}}^{1} |f|^r d\lambda' + \int_{\substack{0 \\ 1<|f|}}^{1} |f|^r d\lambda'$$

we are using the fact that $\mu([0,1]) < \infty$

$r<s$

$$\le 1 + \int_{\substack{0 \\ |f|>1}}^{1} |f|^s d\lambda' \le 1 + \int_0^1 |f|^s d\lambda'$$

therefore $\int_0^1 |f|^s d\lambda' < \infty \Rightarrow \int_0^1 |f|^r d\lambda' < \infty$. /

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Note that it was important for our Proof that $\mu([\cdot,1]) < \infty$

and in this particular case is equal to one.

ex. $\int_1^\infty \left|\frac{1}{x}\right|$ diverges but $\int_1^\infty \left|\frac{1}{x}\right|^2 < \infty$!

Also

f^s μ-integrable does not necessarily implies that

f^r is μ-integrable for $1 \le r < s$. even if the whose

space has finite measure. bcz it can happen that f^s is

measurable but f^r is not.

ex. let $A \subset [\cdot,1]$ be a non-measurable subset and

$$f(x) = \begin{cases} 1 & \text{if } x \in A \\ -1 & \text{"} \ x \notin A \end{cases}$$

it is not measurable but we have: that f^2 is

λ'-meas. and $\int_{[\cdot,1]} |f| \, d\lambda' < \infty$ but f is not μ-integrable

bcz f is not measurable. /