- Tensor is a linear map -- $\begin{cases} S(u+v) = S(u) + S(v) \\ S(\alpha u) = \alpha S(u) \end{cases}$

- Let S and T be two vectors :

$$S = T \iff Sv = Tv \quad \forall v$$

- The space of all tensors is a vector space :

$$\begin{cases} (S+T)(v) = S(v) + T(v) \\ (\alpha S)(v) = \alpha S(v) \end{cases}$$

- The tensor product of two vectors a and b can define the tensor: $(a \otimes b)(v) = (b \cdot v) a \quad \forall v$

$\longrightarrow$ Problem $\underline{2}$.

Let e, f be orthogonal unit vectors in R^3.

(i) show that $(\mathbb{1} - e \otimes e) v = - e \times (e \times v)$.

(ii) What does $(\mathbb{1} - e \otimes e)$ mean geometrically?

(iii) " " $e \otimes e + f \otimes f$ mean " ?

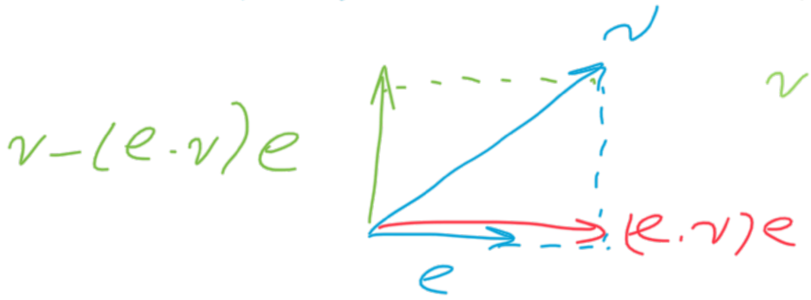
Solution :

(i) & (ii) : $\big(\mathbb{1} - (e \otimes e)\big) v = v - (e \otimes e) v$

$$= v - \underbrace{(e \cdot v) e} = \text{LHS}$$

This is the projection of v on e

the projection of v onto the plane Perpendecular to e.



RHS : $-e \times (e \times v) = -(e \cdot v) e + (e \cdot e) v$

$$= v - (e \cdot v) e$$

$\Rightarrow$ RHS = LHS ✓

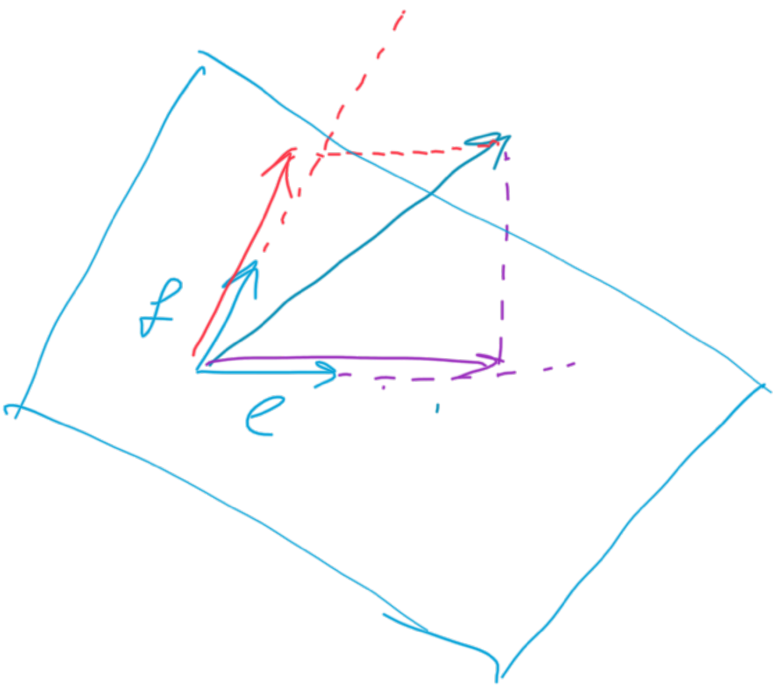
(iii)

$$(e \otimes e + f \otimes f) v = (e \otimes e) v + (f \otimes f) v$$

$$= \underbrace{(e \cdot v) v} + \underbrace{(f \cdot v) f}$$

Projection of v onto e

projection of v onto f

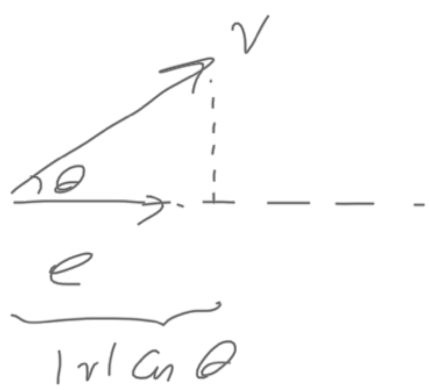


Therefore, the sum is the projection of v onto the plane spanned by e and f.

Ex1: What is the geometrical meaning of the tensor $e \otimes e$ where e is a unit vector in $\mathbb{R}^3$?

$$(e \otimes e)(v) = (e \cdot v) \cdot e$$

The Projection of v onto e.



$$|(e \cdot v) e| = |e \cdot v|$$

$$= |e| |v| \cos\theta = |v| \cos\theta$$

Ex2: Show that $u \times (v \times w) = [(w \cdot u)\mathbb{1} - w \otimes u] v$.

HW 9 _ Problem 8 _ (ii)

$$u \times (v \times w) = \underbrace{(u \cdot w) v}_{(w \cdot u) v} - \underbrace{(u \cdot v) w}_{\substack{(w \otimes u) v \\ \text{by def.}}} =$$

$$= \overbrace{(w \cdot u)}^{\text{Scalar}} v - \overbrace{(w \otimes u)}^{\text{tensor}} v$$

$$= \Big(\underbrace{(w \cdot u)\mathbb{1} - (w \otimes u)}_{\text{tensor}} \Big) v$$

- Let e_1, e_2, e_3 be the basis of $\mathbb{R}^3$, then the components of a tensor T are: $T_{ij} = e_i \cdot T e_j$

- Every tensor can be written of the form:

$$T = T_{ij} (e_i \otimes e_j)$$

- Transpose of a tensor $S : V \longrightarrow V$ is the tensor S^T s.t.

$$v \cdot S w = w \cdot S^T v \quad \forall v, w \in V$$

$$\begin{cases} S = S^T \longleftrightarrow S \text{ is sym.} \\ S = -S^T \longleftrightarrow S \text{ is skew-sym. (Anti-sym.)} \end{cases}$$

$$\left.\begin{aligned} \mathrm{sym}(S) &= \frac{1}{2}(S + S^T) \\ \mathrm{skew}(S) &= \frac{1}{2}(S - S^T) \end{aligned}\right\} \oplus \rightarrow S$$

- product of two tensors $S, T : V \longrightarrow V$ is the composition:

$$(ST)(v) = S(T(v)) \quad \forall v$$

Ex 3. Show that $(ST)^T = T^T S^T$

$$u \cdot (ST)^T v = (ST)u \cdot v$$
$$= S(Tu) \cdot v$$
$$= Tu \cdot S^T v$$
$$= u \cdot T^T S^T v$$

$\forall u, v$. Since transpose is unique $(ST)^T = T^T S^T$.

$\boxed{\#}$

$\longrightarrow$ Problem $\underline{3}$ (use Ex 3.)

show that $\mathrm{sym}(A^T B A) = A^T \mathrm{sym}(B) A$

for A, B tensors $R^d \longrightarrow R^d$.

Solution:

$$\mathrm{sym}(A^T B A) = \frac{1}{2}\left(A^T B A + (A^T B A)^T\right)$$

$$\overset{Ex3}{=} \frac{1}{2}\left(A^T B A + A^T B^T A\right) = \frac{1}{2}\left(A^T (B + B^T) A\right)$$

$$= A^T \left(\frac{1}{2}(B + B^T)\right) A = A^T \mathrm{sym}(B) A.$$

Prop 2.2 (and proof.).

$$(ST)_{ij} = S_{ik} T_{kj}$$

- Trace of a tensor: $tr(a \otimes b) := a \cdot b \quad \forall a, b$

$Tr\, S = S_{ii}$

→ Problem 1.

Show that $tr(ST) = tr(TS)$

for S, T tensors $V \longrightarrow V$ where V is a finite dim. Vector Space.

Solutions: $tr(ST) = (ST)_{ii} = S_{ik} T_{ki}$

$$= T_{ki} S_{ik}$$

$$= (TS)_{kk} = tr(TS).$$

$\boxed{\#}$

- Inner product of tensors: $S : T := tr(S^T T)$

$$S : T = tr(S^T T) \overset{*}{=} tr(T^T S) = S_{ij} T_{ij}$$

$$S^T_{ij} T_{ji} = S_{ji} T_{ji} = T_{ji} S_{ji} = T^T_{ij} S_{ji} = tr(T^T S)$$

$\longrightarrow$ problem $\underline{5}$.

- Show that $S : W = 0$ for S, W tensors $\mathbb{R}^d \longrightarrow \mathbb{R}^d$ where S is symmetric and T is antisymmetric (skew-symmetric).

Solution:

$$S : W = tr(S^T W)$$

(from * and 7) $= tr(W^T S)$

(from problem 7) $= tr(S W^T)$

(W is skew-sym) $= -tr(S W)$

(S is sym) $= -tr(S^T W)$

$$= -S : W$$

$\Longrightarrow S : W = 0$. ▣

- $|S|^2 = S : S$

Every skew tensor can be written of the form $(\omega \times)$ for some $\omega \in \mathbb{R}^3$ as you prove in Problem 5(i).
ω is called the axial vector of Ω.

Ex 4: Let Ω be a skew tensor of the form

$$\Omega(v) = \omega \times v \quad \forall v \in \mathbb{R}^3$$

show that

$$|\Omega|^2 = 2|\omega|^2 .$$

$$|\Omega^2| = \Omega : \Omega = \operatorname{tr}(\Omega^T \Omega)$$
$$= \operatorname{tr}(-\Omega\Omega)$$
$$= -\operatorname{tr}(\Omega\Omega) \quad *$$

$$\Omega(\Omega(v)) = \omega \times (\omega \times v) \overset{HW9\_P8\text{-}ii}{=} (\omega \cdot v)\omega - (\omega \cdot \omega) v$$
$$= (\omega \otimes \omega)(v) - |\omega|^2 v$$
$$= \left((\omega \otimes \omega) - |\omega|^2 \mathbb{1}\right) v$$

$$* = -\operatorname{tr}\left((\omega \otimes \omega) - |\omega|^2 \mathbb{1}\right)$$

$\operatorname{tr}(\alpha S + \beta T) = \alpha \operatorname{tr}(S) + \beta \operatorname{tr}(T)$

$$= -\operatorname{tr}(\omega \otimes \omega) + |\omega|^2 \operatorname{tr} \mathbb{1} = -\omega \cdot \omega + 3|\omega|^2$$
$$= -|\omega|^2 + 3|\omega|^2 = 2|\omega|^2 .$$

- A tensor S is invertible if there exists a tensor T s.t. $ST = TS = \mathbb{1}$.

- Determinant: $$\frac{Su\cdot(Sv\times Sw)}{u\cdot(v\times w)} := \det(S)$$

$$S \text{ is invertible} \iff \det S \neq 0$$

Proof in Gurtinetals P 21.

- Cofactor of a tensor S:

$$S^c = (\det S)\, S^{-T} \qquad S^c(u\times v) = S(u)\times S(v) \quad (*)$$

$\forall u, v$ linearly independent.

Ex 5. Show that for $u\cdot v \neq -1$, the tensor $T = \mathbb{1} + u\otimes v$ is invertible:

① show that the tensor

$$S = \mathbb{1} - (1 + u\cdot v)^{-1}\, u\otimes v$$

satisfies $TS = ST = \mathbb{1}$

OR

② show that $\det T \neq 0$.

Any of ① or ② proves that T is invertible.

① $ST(w) = S\big((\mathbb{1} + u\otimes v)w\big)$

$$= S\big(w + (u\otimes v)w\big) = S\big(w + (v\cdot w)u\big)$$

$$= S(w) + (v\cdot w)\,S(u)$$

$$= \Big(\big(\mathbb{1} - (1+u\cdot v)^{-1}(u\otimes v)\big)(w)$$

$$+ (v\cdot w)\big(\mathbb{1} - (1+u\cdot v)^{-1}(u\otimes v)\big)(u)\Big)$$

$$= w - \frac{(v\cdot w)u}{1 + u\cdot v}$$

$$+ (v\cdot w)\left(u - \frac{(v\cdot u)\,u}{1+u\cdot v}\right)$$

$$= w + \frac{\cancel{-((v\cdot w)u)(1+u\cdot v) + (v\cdot w)u(1+u\cdot v)} - (v\cdot w)(v\cdot u)u}{1+uv} \to 0$$

$$= w \implies ST = \mathbb{1}$$

Similarly you can show that $TS = \mathbb{1}$.

② WTS that $\det T \neq 0$.

$$\det(T) = \frac{T(w_1)\cdot(T(w_2)\times T(w_3))}{w_1\cdot(w_2\times w_3)}$$

for some $w_1, w_2, w_3 \in \mathbb{R}^3$ linearly indep.

$$\det(T) = \det(\mathbb{1} + u\otimes v) =$$

$$\frac{[(\mathbb{1}+u\otimes v)(w_1)]\cdot[((\mathbb{1}+u\otimes v)(w_2))\times((\mathbb{1}+u\otimes v)(w_3))]}{w_1\cdot(w_2\times w_3)}$$

$$= \frac{[w_1 + (v\cdot w_1)u]\cdot[(w_2+(v\cdot w_2)u)\times(w_3+(v\cdot w_3)u)]}{w_1\cdot(w_2\times w_3)}$$

$$= \frac{\overbrace{[w_1+(v\cdot w_1)u]\cdot[w_2\times w_3 + (v\cdot w_2)u\times w_3 + (v\cdot w_3)w_2\times u]}^{A}}{w_1\cdot(w_2\times w_3)}$$

$$A = w_1\cdot(w_2\times w_3) + (v\cdot w_2)\underbrace{w_1\cdot(u\times w_3)}_{u\cdot(w_3\times w_1)} + (v\cdot w_3)\underbrace{w_1\cdot(w_2\times u)}_{u\cdot(w_1\times w_2)}$$

$$+ (v\cdot w_1)u\cdot(w_2\times w_3)$$

let $w_1 = e_1$, $w_2 = e_2$, $w_3 = e_3$

$$A = 1 + v_2u_2 + v_3u_3 + v_1u_1 = 1 + u\cdot v$$

$\det(T) = 1 + u\cdot v \neq 0$ since $u\cdot v \neq -1$ ∎

→ Problem 4

Let Ω be a skew-sym. tensor $\mathbb{R}^3 \to \mathbb{R}^3$.

(i) Show that $\exists w \in \mathbb{R}^3$ s.t.:

$$\forall v \in \mathbb{R}^3 \quad \Omega(v) = w \times v$$

(ii) Show that cofactor of Ω is $w \otimes w$.

(iii) Prove that $\mathbb{1} + \Omega$ is invertible.

For (i) write the components of the both sides and use the ε-δ identity (from HW9) to find the proper vector w.

For (iii) show that the determinant is nonzero

For (ii) show that $(w \otimes w)$ has the property $(*)$.

Solution:

(i) we need to show that $\exists w \in \mathbb{R}^3$ s.t.

$$\Omega_{kj} = (\omega\times)_{kj}$$

where $\omega\times$ is a tensor $\mathbb{R}^3 \to \mathbb{R}^3$ s.t.

$$(\omega\times)u = \omega \times u := v$$

$$v_k = (\omega \times u)_k = \omega_i u_j \varepsilon_{ijk} \Rightarrow v_k = (\varepsilon_{ijk}\omega_i) u_j$$

NOTE that from the def $T_{ij} = e_i \cdot T e_j$

Therefore, if $T(u) = v \Rightarrow T(u_i e_i) = v_k e_k$

$\Rightarrow u_i T(e_i) = v_k e_k \Rightarrow v_k = u_i (e_k \cdot T(e_i))$

$= u_i T_{ki}$

$\Rightarrow v_k = (\omega \times)_{kj} u_j$ where $\boxed{(\omega \times)_{kj} = \varepsilon_{ijk} \omega_i}$ $(*)$

$$\Omega_{kj} = \frac{1}{2}\left(\Omega_{kj} - \Omega^T_{kj}\right)$$

$$= \frac{1}{2}\left(\Omega_{kj} - \Omega_{jk}\right)$$

$$= \frac{-1}{2}\left(\delta_{pj}\delta_{qk} - \delta_{pk}\delta_{qj}\right)\Omega_{pq}$$

$\epsilon - \delta$ identity from HW9 p.8 : $\epsilon_{ijk} \epsilon_{ipq} = \delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}$

$$= \frac{-1}{2}\left(\varepsilon_{ipq}\varepsilon_{ijk}\right)\Omega_{pq} \quad (**)$$

Setting $(*) = (**)$

$$\cancel{\varepsilon_{ijk}}\, \omega_i = \frac{-1}{2} \varepsilon_{ipq} \cancel{\varepsilon_{ijk}}\, \Omega_{pq}$$

$$\Rightarrow \omega_i = \frac{1}{2} \varepsilon_{piq} \Omega_{pq} \,. \checkmark$$

(iii) WTS $\det(\mathbb{1} + \Omega) \neq 0$ where Ω is $(\omega \times)$ for some $\omega \in \mathbb{R}^3$.

$$\det(\mathbb{1} + \Omega) = \det(\mathbb{1} + \omega \times) = \frac{(\mathbb{1} + \omega \times)(e_1) \cdot \left((\mathbb{1} + \omega \times)e_2 \times (\mathbb{1} + \omega \times)e_3\right)}{e_1 \cdot (e_2 \times e_3)}$$

$$= (e_1 + \omega \times e_1) \cdot \left((e_2 + \omega \times e_2) \times (e_3 + \omega \times e_3)\right)$$

$$= (e_1 + \omega \times e_1) \cdot \Big(\overbrace{e_2 \times e_3}^{e_1} + \overbrace{e_2 \times (\omega \times e_3) + (\omega \times e_2) \times e_3}^{\text{use HW9-P8 (iii)}}$$

$$+ \underbrace{(\omega \times e_2) \times (\omega \times e_3)}\Big)$$

use (and prove) $(a \times b) \times (a \times c) = (a \cdot (b \times c))a$

$$= (e_1 + \omega \times e_1) \cdot \left(e_1 + \overbrace{\omega \times e_1} + \overbrace{(\omega \cdot e_1)\,\omega}\right)$$

$$= 1 + (\omega \cdot e_1)^2 + \underbrace{(\omega \times e_1) \cdot (\omega \times e_1)}_{\sim}$$

HW9-P8 (ii)

$$= 1 + (\omega \cdot e_1)^2 + \omega \cdot \underbrace{(e_1 \times (\omega \times e_1))}$$

$$= 1 + (\omega \cdot e_1)^2 + \omega \cdot \left(\overbrace{(e_1 \cdot e_1)\,\omega - (e_1 \cdot \omega)\,e_1}\right)$$

$$= 1 + \cancel{(\omega \cdot e_1)^2} + |\omega|^2 - \cancel{(\omega \cdot e_1)^2} = 1 + |\omega|^2 \neq 0$$

∎

(ii)

WTS $\Omega u \times \Omega v = (\omega \otimes \omega)(u \times v)$

$$\Omega u \times \Omega v = (\omega \times u) \times (\omega \times v)$$

$$= ((\omega \times u) \cdot v)\,\omega - \overbrace{((\omega \times u) \cdot \omega)}^{0}\,v$$

p8 (ii)

$$= (\omega \cdot (u \times v))\,\omega$$

$$= (\omega \otimes \omega)(u \times v) \quad \checkmark$$

A brief introduction to permutations:

Permutation is a bijection

$$\{1,2,3,\dots,n\} \longrightarrow \{1,2,3,\dots,n\}$$

- write down all the permutations on $\{1,2\}$

$$\pi_1: \begin{cases} 1 \longrightarrow 1 \\ 2 \longrightarrow 2 \end{cases} \qquad \pi_2: \begin{cases} 1 \longrightarrow 2 \\ 2 \longrightarrow 1 \end{cases}$$

we denote them as $\pi_1 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ and $\pi_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

- write down all the permutations on $\{1,2,3\}$

$$\pi_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \qquad \operatorname{sgn}(\pi_1) = 1$$

$$\pi_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \overbrace{(2\ 3)}^{-1}\cancel{(1)} \qquad \operatorname{sgn}(\pi_2) = -1$$

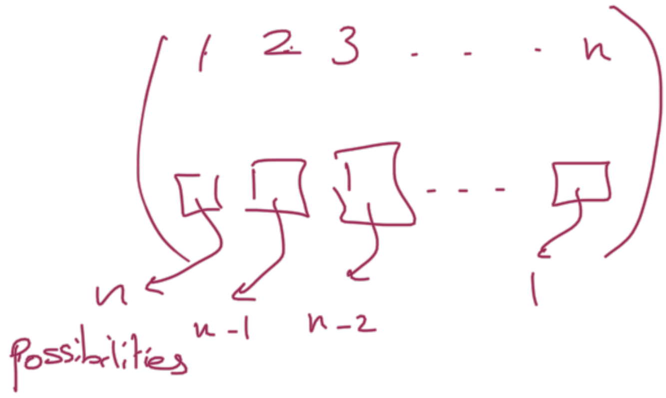
$$\pi_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \overbrace{(1\ 2)}^{-1}\cancel{(3)} \qquad \operatorname{sgn}(\pi_3) = -1$$

$$\pi_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \overbrace{(1\ 2\ 3)}^{1} \qquad \operatorname{sgn}(\pi_4) = 1$$

$$\pi_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \overbrace{(1\ 3\ 2)}^{1} \qquad \operatorname{sgn}(\pi_5) = 1$$

$$\pi_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \overbrace{(1\ 3)}^{-1}\cancel{(2)} \qquad \operatorname{sgn}(\pi_6) = -1$$

Question: How many permutations does $\{1, 2, \cdots, n\}$ have?



$n!$ Permutations

The set of all permutations on $\{1, \cdots, n\}$ is denoted by S_n.

Every permutation can be written as composition of cycles.

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{pmatrix} = \overbrace{(1\ 3)}^{\sigma_1}\overbrace{(2\ 4\ 5)}^{\sigma_2}$$

Sign of a cycle is positive if the length of the cycle is odd and is negative if the length is even. ($\text{sgn}^{**}(\text{identity}) = 1$)

Ex: $\text{Sgn}\, \sigma_1 = -1$ & $\text{sgn}\, \sigma_2 = +1$

Sign of a permutation is the multiplication of signs of its cycles.

Ex: $\text{Sgn}\, \pi = (\text{Sgn}\, \sigma_1)(\text{sgn}\, \sigma_2) = -1 \times +1 = \boxed{-1}$

find the sign of all permutations of $\{1, 2, 3\}$.

Now you can define the determinant as follows:

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

$$\det A = \sum_{\pi \in S_n} \operatorname{sgn}(\pi)\, a_{1\pi(1)}\, a_{2\pi(2)} \cdots a_{n\pi(n)}.$$

for 3x3 matrix:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$\overbrace{(1\ 2)}^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

$\operatorname{sgn}(\pi_1)$ $\operatorname{sgn}(\pi_2)$ $\operatorname{sgn}(\pi_3)$

$$(+1)\, a_{11} a_{22} a_{33} + (-1)\, a_{11} a_{23} a_{32} + (-1)\, a_{12} a_{21} a_{33}$$

$\operatorname{sgn}(\pi_4)$ $\operatorname{sgn}(\pi_5)$ $\operatorname{sgn}(\pi_6)$

$$+ (+1)\, a_{12} a_{23} a_{31} + (-1)\, a_{13} a_{22} a_{31} + (+1)\, a_{13} a_{21} a_{32}$$

$$a_{1\pi_6(1)}\ a_{2\pi_6(2)}\ a_{3\pi_6(3)}$$

Remember: $(fg)' = f'g + g'f$

Therefore,

$$(f_1 f_2 \cdots f_n)' = f_1' f_2 \cdots f_n + f_1 f_2' \cdots f_n + \cdots + f_1 f_2 \cdots f_n'$$

Problem 6, HW-10: WTS

$$(\det F)' = \sum_{i=1}^{n} \det \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \cdots & \cdots & & \cdots \\ f_{i1}' & f_{i2}' & \cdots & f_{in}' \\ \cdots & \cdots & & \cdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix}$$

Set $n=2$, what does the problem mean?

$$F = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$$

we want to show

$$(f_{11} f_{22} - f_{12} f_{21})' = \det \begin{pmatrix} f_{11}' & f_{12}' \\ f_{21} & f_{22} \end{pmatrix} + \det \begin{pmatrix} f_{11} & f_{12} \\ f_{21}' & f_{22}' \end{pmatrix}$$

$$RHS = f_{11}' f_{22} - f_{12}' f_{21} + f_{11} f_{22}' - f_{12} f_{21}'$$

$$LHS = (f_{11} f_{22})' - (f_{12} f_{21})' =$$

$$f_{11}' f_{22} + f_{11} f_{22}' - f_{12}' f_{21} - f_{12} f_{21}'$$

$\Rightarrow RHS = LHS$. ✓

Now you can use the definition of determinant using permutations:

$$(\det F) = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) f_{1\pi(1)} f_{2\pi(2)} \cdots f_{n\pi(n)}$$

for $n=2$ $\quad \det F = \operatorname{sgn}(\pi_1) f_{1\pi_1(1)} f_{2\pi_1(2)}$

$$+ \operatorname{sgn}(\pi_2) f_{1\pi_2(1)} f_{2\pi_2(2)}$$

where $\pi_1 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ and $\pi_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

$$\Rightarrow (\det F)' = (\operatorname{sgn} \pi_1)\left(f'_{1\pi_1(1)} f_{2\pi_1(2)} + f_{1\pi_1(1)} f'_{2\pi_1(2)} \right)$$

$$+ (\operatorname{sgn} \pi_2)\left(f'_{1\pi_2(1)} f_{2\pi_2(2)} + f_{1\pi_2(1)} f'_{2\pi_2(2)} \right)$$

$$= (\operatorname{sgn} \pi_1)\left(f'_{1\pi_1(1)} f_{2\pi_1(2)} \right) + (\operatorname{sgn} \pi_2)\left(f'_{1\pi_2(1)} f_{2\pi_2(2)} \right)$$

$$+ (\operatorname{sgn} \pi_1)\left(f_{1\pi_1(1)} f'_{2\pi_1(2)} \right) + (\operatorname{sgn} \pi_2)\left(f_{1\pi_2(1)} f_{2\pi_2(2)} \right)'$$

$$= \sum_{\pi \in S_2} f'_{1\pi(1)} f_{2\pi(2)} + \sum_{\pi \in S_n} f_{1\pi(1)} f'_{2\pi(2)}$$

$$= \det \begin{pmatrix} f'_{11} & f'_{12} \\ f_{21} & f_{22} \end{pmatrix} + \det \begin{pmatrix} f_{11} & f_{12} \\ f'_{21} & f'_{22} \end{pmatrix}.$$

Solution to problem 6_ HW10.

Let $m = n!$

$$\det F = \sum_{\pi \in S_n} (\operatorname{sgn} \pi) f_{1\pi(1)} f_{2\pi(2)} \cdots f_{n\pi(n)}$$

$$= \operatorname{Sgn}(\pi_1) f_{1\pi_1(1)} f_{2\pi_1(2)} \cdots f_{n\pi_1(n)}$$

$$+ \operatorname{Sgn}(\pi_2) f_{1\pi_2(1)} f_{2\pi_2(2)} \cdots f_{n\pi_2(n)}$$

$$\vdots$$

$$+ \operatorname{Sgn}(\pi_m) f_{1\pi_m(1)} f_{2\pi_m(2)} \cdots f_{n\pi_m(n)}$$

$$\Longrightarrow$$

$$
\begin{aligned}
(\operatorname{Det} F)' = \operatorname{Sgn}(\pi_1) \Big(& f'_{1\pi_1(1)} f_{2\pi_1(2)} \cdots f_{n\pi_1(n)} \\
& + f_{1\pi_1(1)} f'_{2\pi_1(2)} \cdots f_{n\pi_1(n)} \\
& \vdots \\
& + f_{1\pi_1(1)} f_{2\pi_1(2)} \cdots f'_{n\pi_1(n)} \Big) \\
+ \operatorname{Sgn}(\pi_2) \Big(& f'_{1\pi_2(1)} f_{2\pi_2(2)} \cdots f_{n\pi_2(n)} \\
& + f_{1\pi_2(1)} f'_{2\pi_2(2)} \cdots f_{n\pi_2(n)} \\
& \vdots \\
& + f_{1\pi_2(1)} f_{2\pi_2(2)} \cdots f'_{n\pi_2(n)} \Big) \\
& \vdots \\
+ \operatorname{Sgn}(\pi_m) \Big(& f'_{1\pi_m(1)} f_{2\pi_m(2)} \cdots f_{n\pi_m(n)} \\
& + f_{1\pi_m(1)} f'_{2\pi_m(2)} \cdots f_{n\pi_m(n)} \\
& \vdots \\
& + f_{1\pi_m(1)} f_{2\pi_m(2)} \cdots f'_{n\pi_m(n)} \Big)
\end{aligned}
$$

$$\Rightarrow (\operatorname{Det} F)' = \operatorname{Sgn}(\pi_1) f'_{1\pi_1(1)} f_{2\pi_1(2)} \cdots f_{n\pi_1(n)}$$

$$+ \operatorname{sgn}(\pi_2) f'_{1\pi_2(1)} f_{2\pi_2(2)} \cdots f_{n\pi_2(n)}$$

$$\vdots$$

$$+ \operatorname{sgn}(\pi_m) f'_{1\pi_m(1)} f_{2\pi_m(2)} \cdots f_{n\pi_m(n)}$$

$$+ \operatorname{Sgn}(\pi_1) f_{1\pi_1(1)} f'_{2\pi_1(2)} \cdots f_{n\pi_1(n)}$$

$$+ \operatorname{Sgn}(\pi_2) f_{1\pi_2(1)} f'_{2\pi_2(2)} \cdots f_{n\pi_2(n)}$$

$$\vdots$$

$$+ \operatorname{Sgn}(\pi_m) f_{1\pi_m(1)} f'_{2\pi_m(2)} \cdots f_{n\pi_m(n)}$$

$$\vdots$$

$$+ \operatorname{Sgn}(\pi_1) f_{1\pi_1(1)} f_{2\pi_1(2)} \cdots f'_{n\pi_1(n)}$$

$$+ \operatorname{Sgn}(\pi_2) f_{1\pi_2(1)} f_{2\pi_2(2)} \cdots f'_{n\pi_2(n)}$$

$$\vdots$$

$$+ \operatorname{sgn}(\pi_m) f_{1\pi_m(1)} f_{2\pi_m(2)} \cdots f'_{n\pi_m(n)}$$

$$= \det \begin{pmatrix} f'_{11} & f'_{12} & \cdots & f'_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & & \ddots & \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{pmatrix} + \cdots + \det \begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & & & \\ f'_{n1} & f'_{n2} & \cdots & f'_{nn} \end{pmatrix}$$

$\blacksquare$