Series 2

1. Let V be a finite-dimensional vector space with two different bases $v_1, \dots, v_n$ and $v'_1, \dots, v'_n$. Let the basis change be denoted by $v'_i = \sum_j C_i^j v_j$.

a) Let $(v_i^* \otimes v_j^*)_{1 \le i,j \le n}$ denote the canonically associated basis of $V^* \otimes V^*$, where

$$v_i^* \otimes v_j^*(v_k, v_l) = \delta_{i,k}\delta_{j,l}\,,$$

and let $(v_i \otimes v_j)_{1 \le i,j \le, n}$ denote the canonically dual basis of $V \otimes V$. Suppose $\beta \in V^* \otimes V^*$ and $h \in V \otimes V$ are expressed with respect to the above bases by

$$\beta \;=\; \sum_{i,j} B_{ij} v_i^* \otimes v_j^*, \quad \text{and} \quad h \;=\; \sum_{i,j} H^{ij} v_i \otimes v_j\,,$$

that is, we have two $n \times n$-matrices B and H. How do they transform as matrices with respect to the basis change C? Use matrix notation B, H, C, C^{-1}, C^t where applicable! *(2 pts)*

b) Suppose now that $T \in \bigotimes^r V \otimes \bigotimes^s V^*$ is an (r, s)-tensor represented with respect to the basis $v_1, \dots, v_n$ by $(T^{i_1 \dots i_r}_{j_1 \dots j_s})_{1 \le i_a, j_b \le n,\, 1 \le a \le r,\, 1 \le b \le s}$. Give the formula in index notation for the basis change with respect to (C_i^j). *(2 pts)*

2. (a) Consider the equation

$$(x + y) \cdot \cosh(x - y) \;=\; 2x\,.$$

Show that there exists $\varepsilon > 0$ and $h\colon (1 - \varepsilon, 1 + \varepsilon) \to \mathbb{R}$ such that $y = h(x)$ is a solution with $h(1) = 1$. Compute $h'(1)$. *(2 pts)*

(b) Let $U(n, \mathbb{C}) = \{\, A \in M(n \times n, \mathbb{C}) \mid A \cdot \overline{A}^t = \mathbf{1} \,\}$, the so-called unitary group. Show that near the identity $\mathbf{1}$ the set $U(n, \mathbb{C})$ can be parametrized by an open subset of the linear space

$$\mathfrak{u}(n, \mathbb{C}) = \{\, B \in M(n \times n, \mathbb{C}) \mid B^t = -\overline{B} \,\}\,.$$

What is its dimension? *(2 pts)*

3. Let $\mathbb{C}P^n := \{\mathbb{C} \cdot v \mid v \in \mathbb{C}^{n+1} \setminus \{0\}\,\}$ where

$$\pi\colon \mathbb{C}^{n+1} \setminus \{0\} \;\to\; \mathbb{C}P^n,$$
$$\pi(v) = \mathbb{C} \cdot v := \{\, \lambda v \mid \lambda \in \mathbb{C} \,\},$$

i.e. $\mathbb{C}P^n$ is the set of all complex 1-dimensional subvectorspaces of $\mathbb{C}^{n+1}$, the so-called *complex lines*. $\mathbb{C}P^n$ is called the n-dimensional **complex projective space**. The element $\mathbb{C} \cdot (z_0, \dots, z_n) \in \mathbb{C}P^n$ with $(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\}$ is denoted by $[z_0 : \dots : z_n]$, which we call *homogeneous coordinates* on $\mathbb{C}P^n$, i.e. $[\lambda z_0 : \dots : \lambda z_n] = [z_0 : \dots : z_n]$ for all $\lambda \in \mathbb{C} \setminus \{0\}$.
Let $\mathbb{C}P^n$ carry the following topology:

$$U \subset \mathbb{C}P^n \text{ is open} \quad \Leftrightarrow \quad \pi^{-1}(U) \subset \mathbb{C}^{n+1} \setminus \{0\} \text{ is open},$$

i.e. it is the largest topology on $\mathbb{C}P^n$ such that π is continuous.

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a) Show that for any topological space X, a map $f\colon \mathbb{C}P^n \to X$ is continuous if and only if $f \circ \pi\colon \mathbb{C}^{n+1} \setminus \{0\} \to X$ is continuous, and a map $g\colon X \to \mathbb{C}P^n$ is continuous exactly if for any $U \subset \mathbb{C}P^n$, s.t. $\pi^{-1}(U)$ is open in $\mathbb{C}^{n+1} \setminus \{0\}$, and for any $x \in X$ with $f(x) \in U$ there exists an open subset $V \subset X$ such that $x \in V$ and $f(V) \subset U$. *(1 pt)*

b) Consider the following subsets $U_i \subset \mathbb{C}P^n$ for $i = 0, \ldots, n$, with

$$U_i := \big\{ [z_0 : \ldots : z_n] \,|\, z_i \neq 0 \big\}$$

and with subset topology. Show that

$$\varphi_i\colon \mathbb{C}^n \to U_i,\ \varphi_i(w_1, \ldots, w_n) := [w_1 : \ldots : w_i : 1 : w_{i+1} : \ldots : w_n]$$

are homeomorphisms. *(1 pt)*

c) Compute $\varphi_i^{-1} \circ \varphi_j$ where defined and show that $\big\{ (U_i, \varphi_i, \mathbb{C}^n) \,|\, i = 0, \ldots, n \big\}$ defines a smooth atlas on $\mathbb{C}P^n$. *(1 pt)*

d) Show that from the above it follows that $\mathbb{C}P^n$ is a smooth manifold of real dimension $2n$. *(1 pt)*

4. a) Consider a smooth function $f\colon U \to \mathbb{R}$, $U \subset \mathbb{R}^n$ open. A point $x \in U$ is called a *critical point* of f if $df(x) = 0$. Let $X, Y \in \mathcal{X}(U)$ be two vector fields on U. Show that, if $x \in U$ is a critical point of f, then $X(Y(f))(x) = Y(X(f))(x)$, and this expression depends only on f and the vectors $X(x), Y(x) \in \mathbb{R}^n$. Hence, we can define $Hf(x)(v, w) := X(Y(f))(x)$ where $X(x) = v$, $Y(x) = w$. Explain why $Hf(x)$ is a symmetric 2-0-tensor on $\mathbb{R}^n$. It is called the Hessian of f at x. *2 pts*

b) Can the Hessian $Hf(x)$ also be defined as a 2-0-tensor, independently of a given basis, if x is not a critical point of f? Prove or give a counterexample. *1 pt*

c) Let $U \subset \mathbb{R}^n$ be open. An operation $\nabla\colon \mathcal{X}(U) \times \mathcal{X}(U) \to \mathcal{X}(U)$, $(X, Y) \mapsto \nabla_X Y$ which is bilinear for $\mathcal{X}(U)$ as a k-vector space, $k = \mathbb{R}, \mathbb{C}$ and which satisfies

$$\nabla_{fX} Y = f \nabla_X Y, \quad \nabla_X (fY) = f \nabla_X Y + X(f) \cdot Y\,,$$

for all $X, Y \in \mathcal{X}(U)$, $f \in C^\infty(U)$, is called a **connection**. It is *not* a 2-0-tensor, because $(\nabla_X Y)(x)$ does not depend only on $X(x)$ and $Y(x)$. Show that, however, for any two connection ∇ and ∇', the difference $\nabla - \nabla'$ is a 2-0-tensor, and that the expression

$$T_\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

is a 2-0-tensor, the so-called **torsion** of ∇. *1 pt*

Hand-In: Practice Session Wednesday Oct. 30