Solutions to HW15.

Problem 1.

Review: Directional derivative of $f: U \to \mathbb{R}^n$ $U \subseteq \mathbb{R}^k$

at a point x_0 in the direction v is

$$\lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0)}{t} = \nabla f(x_0) v$$

So all you need is compute $\nabla f(x_0)$ which you know how from HW14. Then, you have a $n \times k$ matrix $\nabla f(x_0)$ and you multiply it by the given vector v.

$$F: U \to \mathbb{R}^2 \qquad U \subseteq \mathbb{R}^3$$

$$(x, y, z) \longmapsto \left(\sin(x+y) - e^z - x \,,\, \ln z - 5x^z \right)$$

$x_0 = (1, -1, 1)$, $v = (1, 0, 3)$.

$$\nabla f = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{bmatrix}$$

$$= \begin{bmatrix} \cos(x+y) - 1 & \cos(x+y) & -e^{z} \\ -5z\,x^{z-1} & 0 & 1/z - 5\ln x . x^{z} \end{bmatrix}$$

substitute x_0 in the matrix ∇f :

$$\Rightarrow \nabla f(x_0) = \begin{bmatrix} 0 & 1 & -e \\ -5 & 0 & 1 \end{bmatrix}$$

Therefore, the directional derivative that we are looking for is:

$$\nabla f(x_0) \cdot v = \begin{bmatrix} 0 & 1 & -e \\ -5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -3e \\ -2 \end{bmatrix}$$

Problem 2.

Review. Arc length parametrization is a parametrization of a curve $\gamma: I \to \mathbb{R}^d$ such that $|\gamma'| = 1$.

For a curve with Arc length parametrization the Frenet frame or TNB frame are:

$$\tau := \gamma' \qquad n := \frac{\gamma''}{|\gamma''|} \qquad b := \tau \times n$$

Therefore, first you need to find a arc-length parametrization of the given curve. and then compute τ, n, b.

$\gamma : [a, b] \to \mathbb{R}^3$

$$\gamma(t) = (\sin t, \cos t, 2t) \Rightarrow \gamma'(t) = (\cos t, -\sin t, 2)$$

$$\Rightarrow |\gamma'(t)| = \sqrt{(\cos^2 t + \sin^2 t + 4)} = \sqrt{5}$$

Let $\tilde{\gamma} : [\sqrt{5}a, \sqrt{5}b] \to \mathbb{R}^3$

$$\tilde{\gamma}(t) = \left(\sin \frac{t}{\sqrt{5}}, \cos \frac{t}{\sqrt{5}}, 2\frac{t}{\sqrt{5}}\right)$$

This is the same curve that now you can see that $|\tilde{\gamma}(t)| = 1$.

$$\tilde{\gamma}'(t) = \left(\frac{1}{\sqrt{5}} \cos \frac{t}{\sqrt{5}}, \frac{-1}{\sqrt{5}} \sin \frac{t}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right)$$

$$|\tilde{\gamma}'(t)| = \sqrt{\frac{1}{5} \cos^2 \frac{t}{\sqrt{5}} + \frac{1}{5} \sin^2 \frac{1}{\sqrt{5}} + \frac{4}{5}} = 1 \checkmark$$

$$\tilde{\gamma}''(t) = \left(-\frac{1}{5} \sin \frac{t}{\sqrt{5}}, \frac{-1}{5} \cos \frac{t}{\sqrt{5}}, 0 \right)$$

$$|\tilde{\gamma}''(t)| = \sqrt{\frac{1}{25} \sin^2 \frac{t}{\sqrt{5}} + \frac{1}{25} \cos^2 \frac{t}{\sqrt{5}}} = \frac{1}{5}$$

$$\Rightarrow n = \frac{\tilde{\gamma}''(t)}{|\tilde{\gamma}''(t)|} = \left(-\sin \frac{t}{\sqrt{5}}, -\cos \frac{t}{\sqrt{5}}, 0 \right)$$

$b = \tau \times n$

$$\begin{bmatrix} i & j & k \\ \frac{1}{\sqrt{5}} \cos \frac{t}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \sin \frac{t}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\sin \frac{t}{\sqrt{5}} & -\cos \frac{t}{\sqrt{5}} & 0 \end{bmatrix}$$

$$b = \left(\frac{2}{\sqrt{5}} \cos \frac{t}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \sin \frac{t}{\sqrt{5}}, \frac{-1}{\sqrt{5}} \underbrace{\left(\cos^2 \frac{t}{\sqrt{5}} + \sin^2 \frac{t}{\sqrt{5}} \right)}_{1} \right)$$

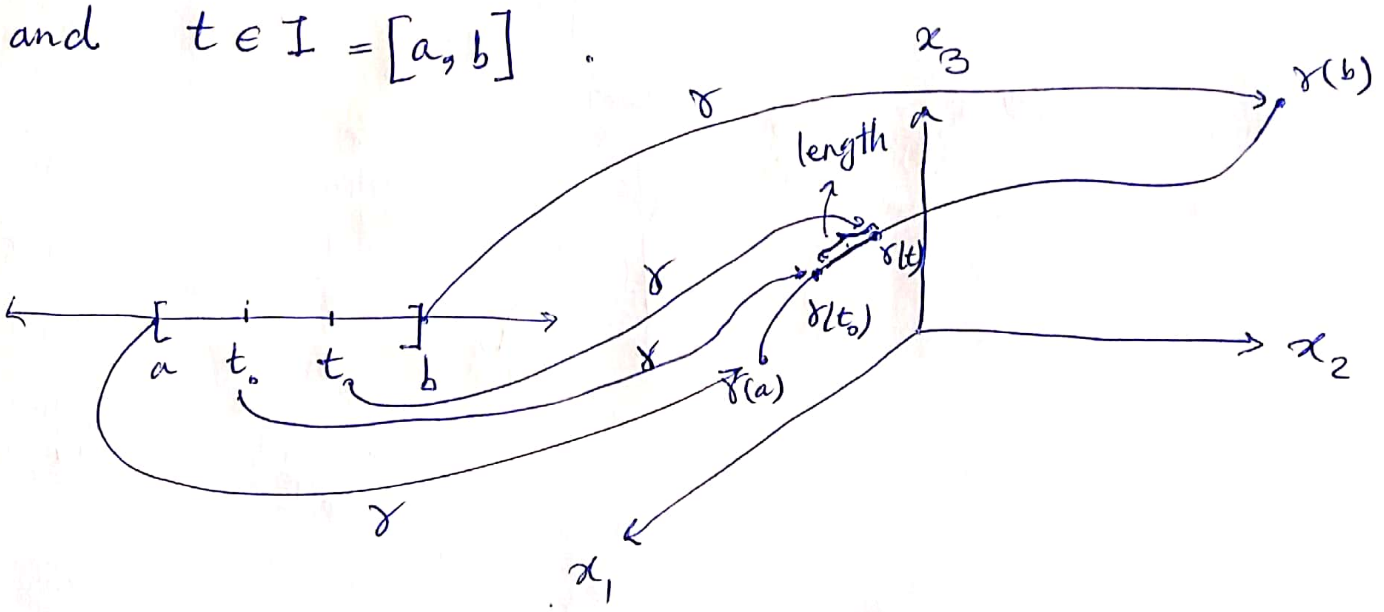
$$= \frac{2}{\sqrt{5}} \left(\cos \frac{t}{\sqrt{5}}, -\sin \frac{t}{\sqrt{5}}, \frac{-1}{2} \right) \checkmark$$

− Example and Explanation on curves in $\mathbb{R}^3$ −

* A curve in $\mathbb{R}^3$ is a function $\gamma : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}^3$

where $\gamma = \gamma(t) = (x_1(t), x_2(t), x_3(t))$

and $t \in I = [a, b]$.



* Length of curve (if rectifiable): $\ell(\gamma(t)) = \int_a^b |\gamma'(t)|\, dt$

* if parameterization by t is regular (i.e $\forall t$ $\gamma'(t) \neq 0$)

then using $$S(t) = \int_{t_0}^{t} |\gamma'(\tau)|\, d\tau$$

we get the length of a piece of the curve between t_0 and t.

Now since $\gamma'(t) \neq 0$ we have $|\gamma'(t)| > 0$

and this implies $S = S(t)$ is an increasing function.

Hence, we can invert the function

and get $\quad t = t(s) \qquad s \in [0, \ell]$

total length of curve

$\Rightarrow$ We can find another parameterization in this manner. (t is not the parameter anymore)

$$\gamma = \gamma(t(s)) = (x_1(t(s)), x_2(t(s)), x_3(t(s)))$$

* By this parameterization: $\ell(\gamma(s)) = s$ since

$$|\gamma'(t)| \underset{\substack{\uparrow \\ \text{chain rule}}}{=} |\gamma'(t(s))| \cdot |s'(t)| \underset{\substack{\uparrow \\ \text{fundamental} \\ \text{theorem of calculus}}}{=} |\gamma'(s(t))| \cdot |\gamma'(t)|$$

$$\frac{d}{dt}\left(s(t) = \int_{t_0}^{t} |\gamma'(t(s))|\, d\tau\right) = \gamma'(t)$$

$$\Rightarrow \quad |\gamma'(s(t))| = 1 \qquad \forall t \in [a,b]$$

Hence $\quad \ell(\gamma(s)) = \int_0^s |\gamma'(\tau)|\, d\tau = s$

Example: cycloid! (curve traced by a moving wheel's tip)

A parameterization by angle is given by

$$\gamma(t) = (a(t-\sin t), a(1-\cos t)) \quad t\in[0,2\pi]$$

$$\gamma'(t) = (a(1-\cos t), a\sin t)$$

$$\Rightarrow S(t) = \int_0^t |\gamma'(\tau)| d\tau$$

$$= \int_0^t \sqrt{a^2(1-\cos t)^2 + a^2\sin^2 t}\, d\tau = a\int_0^t \sqrt{1-2\cos\tau+\cos^2\tau+\sin^2\tau}\, d\tau$$

$$= a\int_0^t \sqrt{2-2\cos\tau}\, d\tau = a\sqrt{2}\int_0^t \sqrt{1-\cos\tau}\, d\tau$$

$$= a\sqrt{2}\int_0^t \sqrt{2\sin^2 \tau/2}\, d\tau = 2a\int_0^t \sin \tau/2 \, d\tau =$$

$$= 4a(1-\cos t/2)$$

$$\Rightarrow \boxed{S(t) = 4a(1-\cos t/2)} \Rightarrow \cos t/2 = 1 - \frac{S(t)}{4a}$$

$$\Rightarrow t/2 = \arccos\left(1 - \frac{S(t)}{4a}\right)$$

$$\Rightarrow \boxed{t(s) = 2\arccos\left(1-\frac{S(t)}{4a}\right)}$$

$$S(2\pi) = 4a(1-\cos\pi) = 8a \quad \Rightarrow S\in[0, 8a]$$

and $\gamma(s) = \gamma(t(s)) = (a(t(s)-\sin t(s)), a(1-\cos(t(s))))$

problem 3.

Thm 4.15. let $F: G \to \mathbb{R}^n$, $\underset{\text{open}}{G \subset \mathbb{R}^{m+n}}$,

F is of class C^1. i.e. differentiable with continuous differential. and rank of $\nabla f(z) = n$

$\forall z \in G$. Then

$$M := \{ z \in G \mid f(z) = 0 \}$$

is a differentiable manifold and $\forall z_0 \in M$

$$T_{z_0} M = \{ h \in \mathbb{R}^{m+n} \mid \nabla F(z_0) h = 0 \}$$

and

$$N_{z_0} M = \text{lin} \{ \nabla F_1(z_0), \dots, \nabla F_n(z_0) \}.$$

let:

$F: \mathbb{R}^3 \to \mathbb{R}$

$$(x, y, z) \longmapsto \frac{x^2}{3} + \frac{y^2}{2} + \frac{z^2}{6} - 5$$

The given set $\left\{ (x, y, z) \in \mathbb{R}^3 : \frac{x^2}{3} + \frac{y^2}{2} + \frac{z^2}{6} = 5 \right\}$

is the zero set of F.

$$\left(\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \quad \frac{\partial f}{\partial z}\right)$$

$$= \left(\frac{2}{3}x \quad y \quad \frac{1}{3}z\right)$$

all the partial derivative exist and they are all continuous therefore F is differentiable and of class C^1.

$z_0 = (1 \ 1 \ 5)$

first observe that $z_0 \in M = \{p \in \mathbb{R}^3 : F(p) = 0\}$

i.e. $\frac{1}{3} + \frac{1}{2} + \frac{25}{6} - 5 = 0$ ✓

Now to find the tangent space at z_0:

$$\nabla f(z_0) = \left(\frac{2}{3}x \quad y \quad \frac{1}{3}z\right)\Big|_{(1 \ 1 \ 5)} = \left(\frac{2}{3} \ 1 \ \frac{5}{3}\right)$$

we want $h = (h_1 \ h_2 \ h_3) \in \mathbb{R}^3$ s.t.

$$\left(\frac{2}{3} \ 1 \ \frac{5}{3}\right) \cdot (h_1 \ h_2 \ h_3) = 0$$

$$\Rightarrow T_{z_0}M = \{h \in \mathbb{R}^3 \text{ s.t. } \underbrace{\tfrac{2}{3}h_1 + h_2 + \tfrac{5}{3}h_3 = 0}\}$$

note that this is the equation of a plane in $\mathbb{R}^3$.

And the normal space to M at z_0:

$$N_{z_0}M = \left\{ t \cdot \left(\frac{2}{3} , \frac{5}{3} \right) : t \in \mathbb{R} \right\}$$

↳ Note that this is a line in $\mathbb{R}^3$.

The line $N_{z_0}M$ is orthogonal to the plane.

Problem 4.

Review. Hessian: let $g : U \to \mathbb{R}$, $U \subseteq \mathbb{R}^n$, $x_0 \in U$

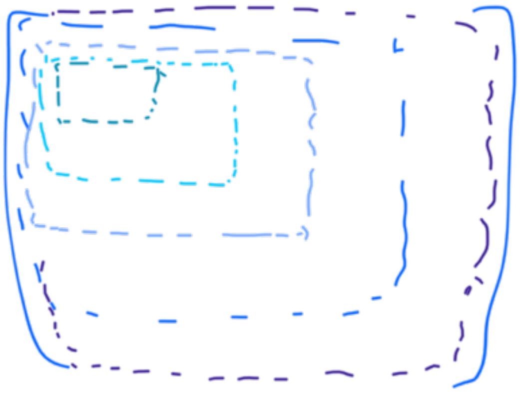
then $Hg(x_0) = \left[\frac{\partial^2 g}{\partial x_k \partial x_j} \right]_{kj}$ $\quad 1 \le k, j \le n$

By Schwarz Theorem (4.16) we know that $Hg(x_0)$ is symmetric since $\frac{\partial^2 g}{\partial x_k \partial x_j} = \frac{\partial^2 g}{\partial x_j \partial x_k}$

$\forall k, j \in \{1, \dots, n\}$

✓

A symmetric matrix is positive definite if the determinant of all the minors of the

Form  (i.e. the $k \times k$ matrices on the top-left corner for $k = 1, \dots, n$) are positive.

It is negative definite if the sign alternates

$\boxed{-}$ $\boxed{+}$ $\boxed{-}$ - - -

Thm 4.17. $g : U \to \mathbb{R}$, $x_0 \in U$,

if

- g is differentiable on U
- It's partial derivatives are differentiable at x_0
- grad $g(x_0) = 0$ (grad $g = \nabla g$ when $g : U \to \mathbb{R}$)

Then

- $Hg(x_0)$ Positive definite $\Rightarrow g(x_0)$ is a local min.
- ~ negative ~ $\Rightarrow$ ~ ~ ~ ~ max.
- $f : \mathbb{R}^2 \to \mathbb{R}$ $\qquad f(x,y) = (x+y)\, e^{-(x^2+y^2)}$

$$\frac{\partial f}{\partial x} = e^{-(x^2+y^2)} + (x+y) \cdot (-2x)\, e^{-(x^2+y^2)}$$

$$\frac{\partial f}{\partial y} = e^{-(x^2+y^2)} + (x+y) \cdot (-2y)\, e^{-(x^2+y^2)}$$

Partial derivatives exist and they are both continuous everywhere on $\mathbb{R}^2$. Therefore, f is differentiable and $\text{grad } f = \nabla f = \left(\frac{\partial f}{\partial x} \ \frac{\partial f}{\partial y}\right)$

we want to know where $\text{grad } f = 0$ so that we can use Thm 4.17.

$$\frac{\partial f}{\partial x} = 0 \rightarrow \quad e^{-(x^2+y^2)}\Big(1 - 2x(x+y)\Big) = 0$$

$$\Rightarrow 2x^2 + 2xy - 1 = 0 \quad (1)$$

$$\frac{\partial f}{\partial y} = 0 \rightsquigarrow 2y^2 + 2xy - 1 = 0 \quad (2)$$

$$(1) - (3) \qquad 2(x^2 - y^2) = 0 \Rightarrow x = \mp y$$

$$\text{In } (1) \begin{cases} y = x & 2x^2 + 2x^2 - 1 = 0 \rightarrow x = \mp \frac{1}{2} \\ y = -x & 2x^2 - 2x^2 - 1 = 0 \ \lightning \end{cases}$$

$P_1 = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $P_2 = \left(\frac{-1}{2}, \frac{-1}{2}\right)$ make $\text{grad } g = 0$.

for these two points we need to check:

- Are the partial derivatives differentiable at P_1 and P_2?

- Are $H g(P_1)$ and $H g(P_2)$ positive or negative definite?

The answer to the first question is clear since the partial derivatives are differentiable everywhere (They are addition and multiplications of known differentiable functions.)

For the second question, let's compute the Hessian:

$$\begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(e^{-(x^2+y^2)}\left(\overbrace{1-2x(x+y)}^{1-2x^2-2xy}\right)\right)$$

$$= -2x\, e^{-(x^2+y^2)} \underbrace{(1-2x(x+y))} + e^{-(x^2+y^2)} \cdot (-4x-2y)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -2y\, e^{-(x^2+y^2)} \underbrace{(1-2x(x+y))} + e^{-(x^2+y^2)}(-2x)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -2y\, e^{-(x^2+y^2)} \underbrace{(1-2y(x+y))} + e^{-(x^2+y^2)} \cdot (-4y-2x)$$

The green expressions are zero for both points P_1 and P_2.

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{(1/2,\,1/2)} = -3e^{-1/2} \qquad \left.\frac{\partial^2 f}{\partial x^2}\right|_{(\frac{-1}{2},\,\frac{-1}{2})} = 3e^{-1/2}$$

$$\left.\frac{\partial^2 f}{\partial y\,\partial x}\right|_{(\frac{1}{2},\,\frac{1}{2})} = -e^{-1/2} \qquad \left.\frac{\partial^2 f}{\partial y\,\partial x}\right|_{(\frac{-1}{2},\,\frac{-1}{2})} = e^{-1/2}$$

$$\left.\frac{\partial^2 f}{\partial y^2}\right|_{(\frac{1}{2},\,\frac{1}{2})} = -3e^{-1/2} \qquad \left.\frac{\partial^2 f}{\partial y^2}\right|_{(\frac{-1}{2},\,\frac{-1}{2})} = 3e^{-1/2}$$

$$Hg(P_1) = \begin{bmatrix} -3e^{-1/2} & -e^{-1/2} \\ -e^{-1/2} & -3e^{-1/2} \end{bmatrix}$$

$-3e^{-1/2} < 0$ and $(-3e^{-1/2})^2 - (-e^{-1/2})^2 = \frac{9}{e} - \frac{1}{e} > 0$

$\Rightarrow g(P_1)$ is a local extrema. (local max)

$$Hg(P_2) = \begin{bmatrix} 3e^{-1/2} & e^{-1/2} \\ e^{-1/2} & 3e^{-1/2} \end{bmatrix}$$

$3e^{-1/2} > 0$ and $e^{-1/2}(9-1) > 0$

$\Rightarrow g(P_2)$ is a local extrema (local min)

- $f: [0,1]^2 \longrightarrow R \quad f(x,y) = x^3 + y^3 - 3xy$

$\frac{\partial f}{\partial x} = 3x^2 - 3y \qquad \frac{\partial f}{\partial y} = 3y^2 - 3x$

Both partial derivatives are continuous everywhere on the domain therefore f is differentiable and

$\text{grad } f = \nabla f = (3x^2 - 3y \quad 3y^2 - 3x)$

In particular, the partial derivatives are differentiable everywhere on the domain.

let grad f = 0 $\begin{cases} x^2 = y \\ y^2 = x \end{cases}$

$y^4 = y \rightarrow y = 0$ or $y^3 = 1 \Rightarrow y = 1$

$\Rightarrow (0,0), (1,1)$ make the gradient zero.

$$Hf = \begin{bmatrix} 6x & -3 \\ -3 & 6y \end{bmatrix}$$

$Hf(0,0) = \begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix}$ ~~ns~~ not positive or negative definite.

$Hf(1,1) = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}$

$6 > 0$ $\quad 36 - (-3)^2 = 36 - 9 > 0$ ✓

$\Rightarrow f(1,1)$ is an extrema (local min)

Problem 5.

we use implicit function theorem.

- $F: G \to \mathbb{R}^n \qquad G \subseteq \mathbb{R}^{m+n}$

- F of class C^1 on G.

- $x_0 \in \mathbb{R}^m, y_0 \in \mathbb{R}^n, (x_0, y_0) \in G, F(x_0, y_0) = 0$

- $\det D_y F(x_0, y_0) \neq 0$

  * The coordinates of x_0 are <u>independent</u> around the point (x_0, y_0)

  * The coordinates of y_0 are <u>dependent</u> around the point (x_0, y_0)

- $D_y f$ is part of the matrix ∇f that is only the partial derivatives wrt coordinates of y. This is a $n \times n$ submatrix of the $n \times (m+n)$ matrix ∇f.

$$\nabla f = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} & \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_n} \\ \vdots & & & & & \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_m} & \frac{\partial f_n}{\partial y_1} & \cdots & \frac{\partial f_n}{\partial y_n} \end{bmatrix}$$

$$\underbrace{\frac{\partial f_n}{\partial x_1} \cdots \frac{\partial f_n}{\partial x_m}}_{D_x f} \quad \underbrace{\frac{\partial f_n}{\partial y_1} \cdots \frac{\partial f_n}{\partial y_n}}_{D_y f}$$

Then,

$\exists f : U \longrightarrow \mathbb{R}^n$ s.t. f is C^1 and

$f(x_0) = y_0$ and $F(x, f(x)) = 0 \quad \forall x \in U$

Although Implicit function thm does not give us the explicit formula for f, we can still compute its differential. (page 70)

$$\nabla f(x_0)_{n \times m} = -\Big(\underbrace{D_y F(x_0, y_0)}\Big)^{-1}_{n \times n} D_x f(x_0, y_0)_{n \times m}$$

Remember that this is a square matrix.

* You can always reorder the variables and move the dependent variables to the end.

$F : \mathbb{R}^3 \longrightarrow \mathbb{R}$

$(x^2+y^2+z^4)^{1/2} - \cos y - \cos z = 0$

$$\frac{\partial F}{\partial y} = \frac{1}{2}(2y)\,(x^2+y^2+z^4)^{-1/2} + \sin y$$

$\frac{\partial F}{\partial y}(2,0,0) = 0 \Rightarrow y$ is independent

around $(2,0,0)$.

$$\frac{\partial F}{\partial x} = \frac{1}{2}(2x)(x^2+y^2+z^4)^{-1/2}$$

$\frac{\partial F}{\partial x}(2,0,0) = 1 \Rightarrow x$ is dependent

around $(2,0,0)$

Thus, there is a function $f : U \longrightarrow \mathbb{R}$ $\quad U \subseteq \mathbb{R}^2$

s.t. $f(0,0) = 2$ and $F(f(y,z), (y,z)) = 0$

$\forall (y,z) \in U$

$$\nabla f(0,0) = \left[\frac{\partial F}{\partial x}(2,0,0)\right]^{-1} \left[\frac{\partial F}{\partial y}(2,0,0) \quad \frac{\partial f}{\partial z}(2,0,0)\right]$$

compute

$$\frac{\partial f}{\partial z} = \frac{1}{2}(4z^3)(x^2+y^2+z^4)^{-1/2} + \sin z$$

Thus,

$$\nabla f(0,0) = 1.\begin{bmatrix} 0 & 0 \end{bmatrix} = 0 \quad ✓$$

Let us check it: (extra)

$$(x^2 + y^2 + z^4) = (\cos y + \cos z)^2$$

$$\rightarrow x^2 = \underbrace{(\cos y + \cos z)^2 - y^2 - z^4}_{A}$$

$$\rightarrow x = \mp \sqrt{A}$$

Take $x > 0$ since we are interested in the nbhd

of $(2,0,0)$

$$\rightarrow x = \left((\cos y + \cos z)^2 - y^2 - z^4\right)^{1/2}$$

$f(y,z) =$ "

check that $\frac{\partial f}{\partial y}(0,0) = 0$ and $\frac{\partial f}{\partial z}(0,0) = 0$. ✓