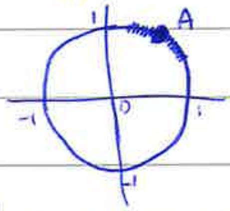
To understand ② b) problem sheet 2:

Let's start with an easy example:

We have a circle



, called C, of radius 1 and origin O, in $\mathbb{R}^2$. When we say we want to parametrize C near A, for example in the nbhd that is thicker, it means that we want to know if there exists a function $h: U \to \mathbb{R}$ ($U \subset \mathbb{R}$, $A \in U$) s.t. the graph of h, i.e. the set of all pts $(x, h(x)) \subset \mathbb{R}^2$ is exactly the part of the circle that is thicker. You can easily see that if you take $h: (-1,1) \to \mathbb{R}$, $x \mapsto \sqrt{1-x^2}$ then the whole upper half of C is the graph of h.

Implicit Function Theorem tells us about the existence of such a function h. It does not give us the explicit formula for it.

An application of implicit function theorem that you have seen is that if you have $M \subset \mathbb{R}^n$ s.t. $0 \in M$, how you can decompose $\mathbb{R}^n$ into $X \oplus Y$ s.t. $\dim X = n-k$ and $\dim Y = k$ for some $1 \leq k < n$, and such that in a nbhd of $0 \in M$, M is the graph of $h: X \supset \mathcal{U}_0 \to Y$, where $\mathcal{U}_0$ is a nbhd of 0 in X.

For this, you need to define M as the level set of some function

$F : M \longrightarrow \mathbb{R}^k$, i.e., $M = \{p \in \mathbb{R}^n$ s.t. $F(p) = y_0$ for some $y_0 \in \mathbb{R}^k\}$.

F has to be in $C^1(U, \mathbb{R}^k)$ for some $\underset{\text{open}}{U \subset M}$ containing 0,

and $DF(0) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^k)$ has to be onto.

Once you have the proper F, the "application" tells you how to

construct $X^{\overset{\ker DF(0)}{=}}$ and Y, such that near 0, M can be the graph
↳ orthogonal to X.

of h, or in other words, near 0 M can be parametrized by an

open subset (here x_0) of X.

what if M does not contain 0, or in general what if we want

to parametrize M near another pt $\overset{\#}{\mathring{x}_0} \in M$.

Easy! You find another function f s.t. $F(p) = f(x_0 + p)$ $\forall p \in \mathbb{R}^n$

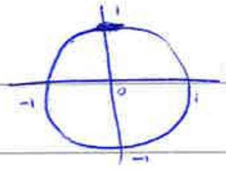
s.t. $x_0 + p \in M$. To be more precise this time let M be a level

set of f i.e. $M = f^{-1}(y_0)$ for a point $y_0 \in \mathbb{R}^k$ so $F(0) = f(x_0) = y_0$

$\Rightarrow 0 \in F^{-1}(y_0)$. Now parametrizing a nbhd of 0 in $F^{-1}(y_0)$ is

the same as parametrizing a nbhd of $x_0 \in f^{-1}(y_0) = M$.

Let's look at our simple example again



we have the circle C and we want to parametrize it in a nbhd of $(1,0)$. So to use Implicit func. thm. let C be the level set of $f: \mathbb{R}^2 \to \mathbb{R}^2$, $(x,y) \mapsto (x^2+y^2-1)$. Hence $C = f^{-1}(0)$

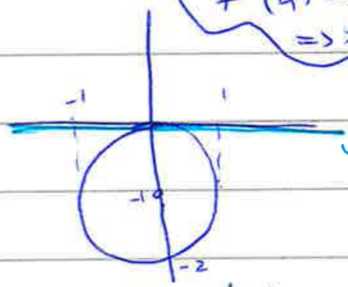
let $F: \mathbb{R}^2 \to \mathbb{R}^2$, $(x,y) \mapsto f(x, y+1)$ $\qquad F((0,0)) = f((0,1)) = 0$

NOTE:
$F(x,y) = f((0,1)+(x,y)) = \alpha$
$\Rightarrow F^{-1}(\alpha) = (x,y)$
$f^{-1}(\alpha) = (0,1) + (x,y)$
$\Rightarrow f^{-1}(\alpha) = (0,1) + F^{-1}(\alpha)$

$DF(0,0) = Df(0,1) = 2\,dx$

$\ker DF(0) = \ker Df(0,1) = \{0\} \times \mathbb{R}$

what is $F^{-1}(0)$



$\hookrightarrow \ker DF(0) = X$

as we see $F^{-1}(0)$ near $(0,0)$ can be parametrized by an open nbhd of $(0,0) \subset X$. But this whole circle $F^{-1}(0)$ is just a shift of C and $(0,1) + \ker DF(0)$ is the right "X" for C, i.e., near $(0,1) \subset C$, C can be characterized by an open subset of $(0,1) + \mathrm{Der}DF(0)$.

In problem ② b) you are looking for a parametrization of the unitary group $U(n, \mathbb{C})$ near the identity $\mathbb{1}$. To use impl. fcn. thm. first you need to find a proper function which has $U(n, \mathbb{C})$ as its level set. That is how we choose

$$f : M(n\times n, \mathbb{C}) \longrightarrow S(n, \mathbb{R}) = \{ B \in M(n\times n, \mathbb{C}) \mid B = \overline{B}^T \}$$
$$A \longmapsto A.\overline{A}^T$$

so $U(n, \mathbb{C}) = f^{-1}(\mathbb{1})$

(↳ * this has nothing to do with the fact that we want to parametrize near $\mathbb{1}$. this could be anything else ....)

Now let $F : M(n\times n, \mathbb{R}) \longrightarrow S(n, \mathbb{R})$ be $F(A) = f(\mathbb{1} + A)$

The reason of choosing $S(n, \mathbb{R})$ is that we want $DF(0)$ to be onto.

• I hope that now the solution makes sense :)

Therefore, near $\mathbb{1}$, $U(n, \mathbb{C})$ can be parametrized by an open subset of the linear space

$$\{ \mathbb{1} + B \mid B \in M(n\times n, \mathbb{C}) \text{ and } \overline{B}^T = -\overline{B} \} .$$

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