5.5 SOLUTION

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Remark 1 It is important to understand the one to one correspondence between $\mathbb{R}$-multilinear maps $T : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{\text{m times}} \times (\mathbb{R}^n)^* \to \mathbb{R}$ and $t : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{\text{m times}} \to \mathbb{R}^n$.

For that first try to see why $(\mathbb{R}^n)^* \otimes \mathbb{R}^n \cong Hom(\mathbb{R}^n, \mathbb{R}^n)$. Let $t \in Hom(\mathbb{R}^n, \mathbb{R}^n)$, $f \in (\mathbb{R}^n)^*$, and $a \in \mathbb{R}^n$, now define $T : \mathbb{R}^n \times (\mathbb{R}^n)^* \to \mathbb{R}$ to be $T(a, f) = (f \circ t)(a)$. Observe that $T \in (\mathbb{R}^n)^* \otimes \mathbb{R}^n$ and this correspondence is one to one.

Before we start with the solution we need to check some notations: For two derivations $X, Y \in \mathcal{X}(M)$ and $f, g \in C^\infty(M)$, by XY we mean the composition of two maps, thus $XY(f) = X(Y(f))$ and by fX we mean the product, i.e., $fX(g) = f.X(g)$. To practice let us see what $[fX, gY]$ is: (By definition $[X, Y] = XY - YX$)

$$\begin{aligned}
[fX, gY](h) &= ((fX)(gY))(h) - ((gY)(fX))(h) \\
&= fX(gY(h)) - gY(fX(h)) \\
&= f.X(g.Y(h)) - g.Y(f.X(h)) \\
&= f\Big(g.X(Y(h)) + Y(h).X(g)\Big) - g\Big(f.(Y(X(h))) + X(h).Y(f)\Big) \\
&= fg(XY(h) - YX(h)) + fX(g)Y(h) - gY(f)X(h)
\end{aligned}$$

Therefore

$$[fX, gY] = fg(XY - YX) + fX(g)Y - gY(f)X \in \mathcal{X}(M) \tag{1}$$

Definition 1 As defined in problem 4.c of sheet 2, a connection $\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$, $\nabla(X, Y) = \nabla_X Y$ is a k-multilinear map of k-vector spaces for $k = \mathbb{C}$ and $\mathbb{R}$, such that

$$\nabla_{fX} Y = f \nabla_X Y \tag{2}$$

$$\nabla_X (fY) = f \nabla_X Y + X(f) Y \tag{3}$$

for all $X, Y \in \mathcal{X}(M)$ and $f \in C^\infty(M)$. (Check my solution 4 sheet 2 for the k-vector space structure of $\mathcal{X}(M)$)

Note: Combining (2) and (3) we have:

$$\nabla_{fX}(gY) = fg \nabla_X Y + fX(g) Y \tag{4}$$

for all $X, Y \in \mathcal{X}(M)$ and $f, g \in C^\infty(M)$.

Now I suggest that before you read the following solution, make sure that you understand 4.c from sheet 2.

Solution 5. To show that $R : \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ is a 3-1-tensor field, we need to show that for any $p \in M$, the map $R(X,Y,Z)_{(p)} := R_p(X(p),Y(p),Z(p))$ is well defined and k-multilinear. R_p being well-defined means that for all $X,Y,Z,X',Y',Z' \in \mathcal{X}(M)$ such that $X(p) = X'(p)$, $Y(p) = Y'(p)$, and $Z(p) = Z'(p)$ we have $R_p(X(p),Y(p),Z(p)) = R_p(X'(p),Y'(p),Z'(p))$. But from 4.c sheet 2 we know that this is enough to show that the map R is k-multilinear and for all $f,g,h \in \mathcal{X}(M)$ we have:

$$R(fX,gY,hZ) = fghR(X,Y,Z) \tag{5}$$

By the definition of R:

$$R(X,Y,Z) := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \tag{6}$$

k-multilinearity of ∇ implies k-multilinearity of R. So it only remains to prove 5.

$$R(fX,gY,hZ) \overset{def}{=} \nabla_{fX}\nabla_{gY}(hZ) - \nabla_{gY}\nabla_{fX}(hZ) - \nabla_{[fX,gY]}(hZ)$$

Now we can separately expand them:

$$\begin{aligned}\nabla_{fX}\nabla_{gY}(hZ) &\overset{(4)}{=} f\nabla_X(gh\nabla_Y Z + gY(h)Z)\\ &\overset{k-lin.}{=} f(\nabla_X(gh\nabla_Y Z) + \nabla_X(gY(h)Z))\\ &\overset{(4)}{=} f\Big(gh\nabla_X(\nabla_Y Z) + X(gh)\nabla_Y Z + gY(h)\nabla_X Z + X(gY(h))Z\Big)\end{aligned}$$

You can simply switch $f \leftrightarrow g$ and $X \leftrightarrow Y$ to see:

$$\nabla_{gY}\nabla_{fX}(hZ) = g\Big(fh\nabla_Y(\nabla_X Z) + Y(fh)\nabla_X Z + fX(h)\nabla_Y Z + Y(fX(h))Z\Big)$$

Finally by using (1) and k-multilinearity of ∇:

$$\begin{aligned}\nabla_{[fX,gY]}(hZ) =& \nabla_{fgXY}(hZ) - \nabla_{fgYX}(hZ) + \nabla_{fX(g)Y}(hZ) - \nabla_{gY(f)X}(hZ)\\ &\overset{(4)}{=}\Big(fgh\nabla_{XY}Z + fg(XY(h))Z\Big) - \Big(fgh\nabla_{YX}Z + fg(YX(h))Z\Big) +\\ &\Big(fX(g)h\nabla_Y Z + fX(g)Y(h)Z\Big) - \Big(gY(f)h\nabla_X Z + gY(f)X(h)Z\Big)\end{aligned}$$

We can now add them up:

$$\begin{aligned}&\Big(fgh\nabla_X(\nabla_Y Z) + fX(gh)\nabla_Y Z + fgY(h)\nabla_X Z + fX(gY(h))Z\Big) -\\ &\Big(fgh\nabla_Y(\nabla_X Z) + gY(fh)\nabla_X Z + fgX(h)\nabla_Y Z + gY(fX(h))Z\Big) -\\ &\Big(fgh\nabla_{XY}Z + fg(XY(h))Z - fgh\nabla_{YX}Z - fg(YX(h))Z +\\ &fX(g)h\nabla_Y Z + fX(g)Y(h)Z - gY(f)h\nabla_X Z - gY(f)X(h)Z\Big)\end{aligned}$$

By simply reordering it you get 5:

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\begin{aligned}
R(fX, gY, hZ) = {} & fgh(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{XY} Z + \nabla_{YX} Z)+ \\
& \Big(fX(gh) - fgX(h) - fX(g)h \Big) \nabla_Y Z- \\
& \Big(gY(fh) - fgY(h) - gY(f)h \Big) \nabla_X Z+ \\
& \Big(fX(gY(h)) - fgX(Y(h)) - fX(g)Y(h) \Big) Z- \\
& \Big(gY(fX(h)) - fgY(X(h)) - gY(f)X(h) \Big) Z
\end{aligned}
$$

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By the definition of a derivation you see that the expression in big parenthesis at each line vanishes. Therefore

$$R(fX, gY, hZ) = fgh(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z).$$