Solutions to the preparatory exercises for HW13

Problem 1

① $\int (xe^x + \sin x)\,dx = \int \underset{u}{\underbrace{x}}\,\underset{v'}{\underbrace{e^x}}\,dx + \int \sin x\,dx$

$= xe^x - \int 1\cdot e^x + \int \sin x\,dx = xe^x - e^x - \cos x$ ✓

② $\int \underset{u}{\underbrace{x}} \cdot \underset{v'}{\underbrace{2^x}}\,dx = x \cdot \frac{2^x}{\ln 2} - \int \frac{2^x}{\ln 2}\,dx$

$= \frac{1}{\ln 2}\, x \cdot 2^x - \frac{1}{\ln 2}\int 2^x\,dx$

$= \frac{1}{\ln 2}\left(x \cdot 2^x - \frac{2^x}{\ln 2}\right)$ ✓

③ $\int \ln x\,dx = \int \underset{v'}{\underbrace{1}} \cdot \underset{u}{\underbrace{\ln x}}\,dx$ (note: $x \in (0,\infty)$)

$= x\ln x - \int x \cdot \frac{1}{x}\,dx = x\ln x - \int 1\,dx$

$= x\ln x - x$

④ $\int \tan^{-1} x \, dx = \int \underbrace{1}_{g'} \cdot \underbrace{\tan^{-1} x}_{f} \, dx$

$= x \tan^{-1} x - \int x \cdot \frac{1}{1+x^2} \, dx$

$= x \tan^{-1} x - \frac{1}{2} \int \frac{2x \, dx}{\underbrace{1+x^2}_{u}}$

$\left(1+x^2 = u \Rightarrow \frac{du}{dx} = 2x \Rightarrow du = 2x \, dx \right)$

$= x \tan^{-1} x - \frac{1}{2} \int \frac{du}{u} = x \tan^{-1} x - \frac{1}{2} \ln u$

$= x \tan^{-1} x - \frac{1}{2} \ln(1+x^2)$ $\quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

⑤ $\int \frac{1}{x^2-5x+6} \, dx = \int \frac{1}{(x-3)(x-2)} \, dx$

$= \int \left(\frac{1}{x-3} - \frac{1}{x-2} \right) dx$

$= \int \frac{1}{x-3} \, dx - \int \frac{1}{x-2} \, dx$

$= \ln(x-3) - \ln(x-2) := G(x)$

$G(x)$ is an anti-derivative function for $\frac{1}{x^2-5x+6}$ on the interval $(3, +\infty)$

Paoblem 2.

(6) $\int_{-3}^{6} |t^2-4|\,dt = \int_{-3}^{-2}(t^2-4)\,dt$

$+ \int_{-2}^{2}(4-t^2)\,dt + \int_{2}^{6}(t^2-4)\,dt$

$= \left[\frac{t^3}{3} - 4t\right]_{-3}^{-2} + \left[4t - \frac{t^3}{3}\right]_{-2}^{2} + \left[\frac{t^3}{3} - 4t\right]_{2}^{6}$

$= \left[(-9-12) - \left(\frac{-8}{3}+8\right)\right] + \cdots = \frac{7}{3} + \frac{32}{3} + \frac{160}{3} = \frac{199}{3}$ ✓

(7) $\int_0^{\pi} \cos(99\theta)\sin(101\theta)\,d\theta$

we use that $\sin(x+y)\cos(x-y) = \frac{1}{2}(\sin 2x + \sin 2y)$ (*)

$= \int_0^{\pi} \cos(100\theta - \theta)\sin(100\theta+\theta)\,d\theta$

$\overset{(*)}{=} \int_0^{\pi} \frac{1}{2}(\sin 200\theta + \sin 2\theta)\,d\theta$

$= \frac{1}{2}\int_0^{\pi} \sin(200\theta)\,d\theta + \frac{1}{2}\int_0^{\pi} \sin 2\theta\,d\theta$

$= \frac{1}{2}\left(-\frac{\cos(200\theta)}{200}\right)\Big]_0^{\pi} + \frac{1}{2}\left(-\frac{\cos(2\theta)}{2}\right)\Big]_0^{\pi}$

$= \frac{-1}{400}\underbrace{\left(\cos(200\pi) - \cos(0)\right)}_{0} - \frac{1}{4}\underbrace{\left(\cos(2\pi) - \cos(0)\right)}_{0} = 0$ ✓

* Comes from the formula

$$(**)\quad \sin\alpha \cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$

by setting $\alpha = x+y$ and $\beta = x-y$

$(**)$ follows from the very known formula:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

now set $\beta = -\beta$: $\cos\beta = \cos(-\beta)$ $\sin\beta = -\sin(-\beta)$ $+$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$(**)$

(8) $$\int_0^{\pi/4} \sec^2(y)\sqrt{\underbrace{2+\tan(y)}}\, dy$$

let $u = \tan y$ then $\frac{du}{dy} = \sec^2(y)$

$\Rightarrow du = \sec^2(y)\,dy$ and $y=0 \to u=0$

$y = \frac{\pi}{4} \to u = 1$

$$= \int_0^1 \sqrt{2+u}\, du = \frac{2}{3}(2+u)^{3/2}\Big]_0^1 =$$

$$\frac{2}{3}\left[(3)^{3/2} - 2^{3/2}\right] = \frac{2}{3}\left(3\sqrt{3} - 2\sqrt{2}\right) = 2\sqrt{3} - \frac{4\sqrt{2}}{3}$$

Problem 3.

Proposition 3.6. Let $f : [1, \infty) \to \mathbb{R}$ be nonnegative and decreasing. Then it holds:

$$\int_1^\infty f \text{ converges} \iff \sum_{i=1}^{\infty} f(i) \text{ converges}$$

9 $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x+1}} \leq \int_1^\infty \frac{dx}{\sqrt{x}}$ because $\displaystyle\frac{1}{\sqrt{x+1}} < \frac{1}{\sqrt{x}}$ $\forall x$.

$\hookrightarrow (**)$

now by Proposition 3.6

$$\int_1^\infty \frac{1}{\sqrt{x}}\, dx \text{ converges} \iff \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ converge } (*)$$

you know that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and

$\forall n \in \mathbb{N},\ \displaystyle\frac{1}{n} \leq \frac{1}{\sqrt{n}} \implies \sum_{n=1}^{\infty} \frac{1}{n} \leq \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

Thus, divergence of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ implies divergence of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.

$\implies$ from $(*)$ $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}$ diverges

and from $(**)$ $\displaystyle\int_1^\infty \frac{1}{\sqrt{1+x}}$ diverges.

Problem 4.

(10) $$\int \frac{\cos x}{\sin x(1-\sin x)}\,dx$$

Let $u = \sin x \to \frac{du}{dx} = \cos x \Rightarrow du = \cos x\,dx$

$$= \int \frac{du}{u(1-u)} = -\int \left(\frac{1}{u-1} - \frac{1}{u}\right) du$$

$$= -\int \frac{du}{u-1} + \int \frac{du}{u} = -\ln|u-1| + \ln|u|$$

$$= \ln|\sin x| - \ln|\sin x - 1|$$

(11) $$\int \frac{2-x}{x^2+1}\,dx = \int\left(\frac{2}{x^2+1} - \frac{x}{x^2+1}\right) dx$$

$$= 2\int \frac{dx}{1+x^2} - \frac{1}{2}\int \frac{\overbrace{2x\,dx}^{du}}{\underbrace{x^2+1}_{u}}$$

$\left(\begin{array}{l} x^2+1 = u \\ 2x\,dx = du \end{array}\right)$

$$= 2\tan^{-1}x - \frac{1}{2}\int \frac{du}{u}$$

$$= 2\tan^{-1}x - \frac{1}{2}\ln|u| = 2\tan^{-1}x - \frac{1}{2}\ln(x^2+1).$$

(12) $\int \sin(\underbrace{\ln x}_{u})\,dx$ 2 methods: I & II

I $\ln x \overset{(*)}{=} u \rightarrow \frac{1}{x} = \frac{du}{dx} \Rightarrow dx = \underbrace{x \cdot du = e^u \cdot du}_{\text{from } (*)}$

$$= \int \underbrace{\sin u}_{f'} \cdot \underbrace{e^u}_{g}\,du = fg - \int fg'$$

$$= (-\cos u)\cdot e^u - \int (-\cos u)e^u\,du$$

$$= -\cos u\, e^u + \int \underbrace{\cos u}_{f'} \cdot \underbrace{e^u}_{g} \cdot du$$

$$= -\cos u\, e^u + \sin u \cdot e^u - \underbrace{\int \sin u \cdot e^u \cdot du}$$

The same expression appeared

$$\Rightarrow 2\int \sin u \cdot e^u \cdot du = e^u \cdot (\sin u - \cos u)$$

$$\Rightarrow \int \sin u \cdot e^u \cdot du = \frac{1}{2} \cdot x\left(\sin(\ln x) - \cos(\ln x)\right)$$

II From the beginning start with integral by parts.

$$\int \sin(\ln x)\,dx = \int \underbrace{1}_{f'} \cdot \underbrace{\sin(\ln x)}_{g}\,dx$$

$$= x \cdot \sin(\ln x) - \int x \cdot \frac{1}{x} \cdot \cos(\ln x)\,dx$$

$$= x \cdot \sin(\ln x) - \int \underbrace{1}_{f'} \cdot \underbrace{\cos(\ln x)}_{g}\,dx$$

$$= x \cdot \sin(\ln x) - \left(x \cos(\ln x) - \int x \cdot \frac{1}{x} \cdot \left(-\sin(\ln x)\right) dx \right)$$

$$= x \sin(\ln x) - x \cos(\ln x) - \int \sin(\ln x)\, dx$$

$$\Longrightarrow 2 \int \sin(\ln x)\, dx = x \left(\sin(\ln x) - \cos(\ln x) \right)$$

$$\Longrightarrow \int \sin(\ln x)\, dx = \frac{1}{2} x \left(\sin(\ln x) - \cos(\ln x) \right) \cdot /$$

(13) $\int \frac{\tan^3(\ln x)}{x}\, dx$

let $u = \ln x \rightarrow du = \frac{1}{x} dx$

$$= \int \tan^3 u\, du = \int \tan^2 u \cdot \tan u \cdot du$$

$$= \int (-1 + \sec^2 u) \tan u \cdot du$$

$v = \sec u \longrightarrow \frac{dv}{du} = \tan u \cdot \sec u \rightarrow dv = \tan u \cdot \overbrace{\sec u}^{v} \cdot du$

$$= \int (v^2 - 1) \cdot \frac{dv}{v} = \int \frac{v^2 - 1}{v} dv = \int v\, dv - \int \frac{1}{v} dv$$

$$= \frac{v^2}{2} - \ln|v| = \frac{\sec^2 u}{2} - \ln|\sec u|$$

$$= \frac{\sec^2(\ln x)}{2} - \ln|\sec(\ln x)| \cdot /$$

Problem 5.

(14) If $\int_2^4 \frac{1}{(x-3)^3}\,dx$ exists then

$$\int_2^4 \frac{1}{(x-3)^3}\,dx = \underbrace{\int_2^3 \frac{1}{(x-3)^3}\,dx}_{A} + \underbrace{\int_3^4 \frac{1}{(x-3)^3}\,dx}_{B}$$

and both A and B must exist. (From Ex1. HW12)

but we can show that $\int_2^3 \frac{1}{(x-3)^3}\,dx$ does not exist. If A exists then let $u = x-3$

$$\int_2^3 \frac{1}{(x-3)^3}\,dx = \int_{-1}^0 \frac{1}{u}\,du = \ln u\Big]_{-1}^0 \;\; ↯$$

This is a contradiction because ln is only defined on $(0, +\infty)$. $\Rightarrow$ (14) does not exist.

(15) $\int_1^\infty \frac{1}{x\ln x}\,dx$

Compute the indefinite integral $\int \frac{1}{x\ln x}\,dx$

Let $\ln x = u \rightarrow \frac{dx}{x} = du$

$$\int \frac{1}{x\ln x}\,dx = \int \frac{du}{u} = \ln u = \ln(\ln x)$$

now look at the boundaries

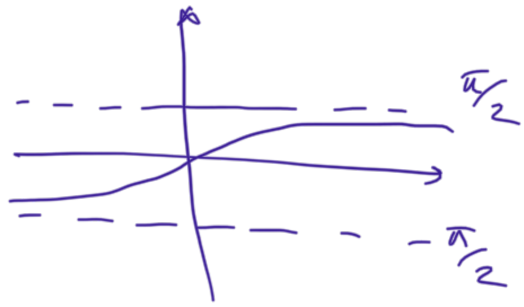
$$\lim_{a \to \infty} \ln(\ln x) \Big]_1^a \longrightarrow \infty$$

$\Rightarrow$ (15) diverges too!

(16) $$\int_0^\infty \frac{1}{4+x^2}\,dx = \frac{1}{2}\int_0^\infty \frac{1/2}{1+\left(\frac{x}{2}\right)^2}\,dx$$

$$= \frac{1}{2}\int_0^\infty \frac{1}{1+u^2}\,du \qquad x/2 = u$$

$$\frac{1}{2}\int \frac{1}{1+u^2}\,du = \frac{1}{2}\tan^{-1} u$$



$$\frac{1}{2}\tan^{-1}\frac{x}{2}\Big]_0^\infty = \pi/4 \qquad \text{Converges!}$$

Problem 6.

(17) $$\int_0^{\pi/2} 4\sin x \cos^3 x$$

$\cos x = u \rightarrow -\sin x\,dx = du$

$$= -4\int_0^{\pi/2} u^3\,du = -4\,\frac{u^4}{4}\Big]_0^{\pi/2} = -u^4\Big]_0^{\pi/2} = -\left(\frac{\pi}{2}\right)^4$$

Since we want the area, we look at the absolute value. $\Rightarrow$ the bounded area $= \left(\frac{\pi}{2}\right)^4$.

Problem 7.

Definition 3.4. Says the length is

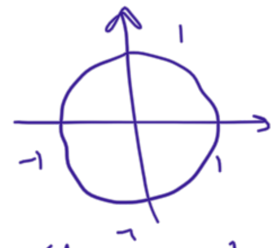
$$\int_a^b \|\gamma'(t)\| \, dt$$

$\gamma(t) = (\sin t, \cos t) \rightsquigarrow \gamma'(t) = (\cos t, -\sin t)$

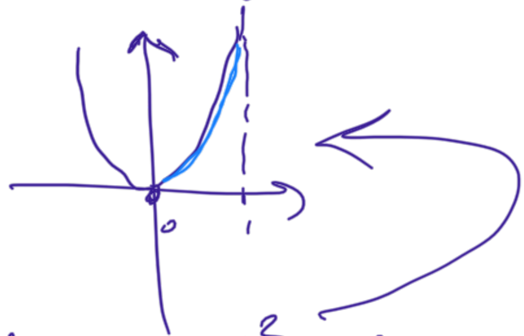
$$\|\gamma'(t)\| = \sqrt{(\cos t)^2 + (-\sin t)^2} = \sqrt{1} = 1$$

$$\int_0^{2\pi} 1 \, dt = t\Big]_0^{2\pi} = \boxed{2\pi}.$$

If you plot $\gamma(t)$ for $t \in [0, 2\pi]$



and you knew it that the length of the unit circle is 2π. :)



Problem 8:

To find the length of $y = \frac{x^2}{2}$ for $0 \leq x \leq 1$

first we need to find the proper parametrisation

$\gamma(t) = (x(t), y(t))$.

setting $x(t) = t$ you get $y(t) = \frac{t^2}{2}$

therefore if $\gamma(t) = \left(t, \frac{t^2}{2}\right)$ $t \in [0,1]$ gives

$\text{im}(\gamma(t))$ = the curve you are interested in.

now again we can use Def 3.4.

$\gamma'(t) = (1, t)$ $\quad \|\gamma'(t)\| = \sqrt{1+t^2}$

$$\int_0^1 (1+t^2)^{1/2} dt =$$

$$\left(\begin{array}{l} t = \tan x \rightarrow dt = \sec^2 x \, dx \\ t=0 \rightarrow x=0 \\ t=1 \rightarrow x = \pi/4 \end{array} \right)$$

$$= \int_0^{\pi/4} (1+\tan^2 x)^{1/2} \sec^2 x \, dx = \int_0^{\pi/4} (\sec^2 x)^{1/2} \sec^2 x \, dx$$

(since $0 \leq x \leq \frac{\pi}{4}$ we have $\sec x > 0 \Rightarrow (\sec^2 x)^{1/2} = \sec x$)

$$= \int_0^{\pi/4} \underbrace{\sec x}_{f} \cdot \underbrace{\sec^2 x \, dx}_{g'}$$

$$= \sec x \cdot \tan x - \int \sec x \cdot \tan x \cdot \tan x \, dx$$

$$= \sec x \cdot \tan x - \int_0^{\pi/4} \sec x \cdot \tan^2 x \, dx$$

$$= \sec x \cdot \tan x - \int_0^{\pi/4} \sec x \cdot (\sec^2 x - 1) \, dx$$

$$= \sec x \cdot \tan x - \int_0^{\pi/4} \sec^3 x \, dx + \int_0^{\pi/4} \sec x \, dx$$

$$\Rightarrow 2 \int_0^{\pi/4} \sec^3 x \, dx = \sec x \cdot \tan x + \underbrace{\int_0^{\pi/4} \sec x \, dx}$$

$$\int \sec x \, dx = \ln |\tan x + \sec x|$$

If you have not seen this before, this is not really easy to find. To check the correctness, check that $\frac{(\tan x + \sec x)'}{\tan x + \sec x} = \frac{1}{\cos x} = \sec x$

$$\Rightarrow \int_0^{\pi/4} \sec^3 x \, dx = \frac{1}{2} \Big[\sec x \tan x + \ln |\sec x + \tan x| \Big]_0^{\pi/4}$$

$$= \frac{1}{2} \Big[\big(\sqrt{2} \cdot 1 + \ln(\sqrt{2} + 1) \big) - \big(0 + \overbrace{\ln(1 + 0)}^{0} \big) \Big)$$

$$= \frac{1}{2} \left(\sqrt{2} + \ln(\sqrt{2} + 1) \right)$$