Review, Hints and Preparatory exercises for HW 15.

Problem 1.

Review: Directional derivative of $f: U \to \mathbb{R}^n$ $U \subseteq \mathbb{R}^k$

at a point x_0 in the direction v is

$$\lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0)}{t} = \nabla f(x_0) v$$

So all you need is compute $\nabla f(x_0)$ which you know how from HW14. Then, you have a $n \times k$ matrix $\nabla f(x_0)$ and you multiply it by the given vector v.

Ex 1: find the directional derivative of the function $f: U \to \mathbb{R}^3$, $U \subseteq \mathbb{R}^2$, defined by

$$f(x,y) = \Big(\overbrace{\cos x - \ln y}^{f_1}, \overbrace{y^x}^{f_2}, e^{\overbrace{xy}^{f_3}}\Big)$$

at the point $\underbrace{(\pi, 1)}_{x_0}$ in the direction $\underbrace{(-2, 3)}_{v}$.

Solution:

$$\nabla f = \begin{bmatrix} -\sin x & -\frac{1}{y} \\ \ln y \cdot y^x & x y^{x-1} \\ y e^{xy} & x e^{xy} \end{bmatrix}$$

$$\nabla f(\pi, 1) \cdot v = \begin{bmatrix} 0 & -1 \\ 0 & \pi \\ e^{\pi} & \pi e^{\pi} \end{bmatrix}_{3\times 2} \begin{bmatrix} -2 \\ 3 \end{bmatrix}_{2\times 1} = \begin{bmatrix} -3 \\ 3\pi \\ e^{\pi}(-2+3\pi) \end{bmatrix}$$

Ex 2. Let $f(x_1,x_2) = \sqrt{|x_1^2 - x_2^2|}$, $(x_1,x_2) \in \mathbb{R}^2$. Determine all directions $\ell \in \mathbb{R}^2$ along which $\frac{\partial f}{\partial \ell}(0,0)$ exists.

Solution: $$\lim_{t \to 0} \frac{f\left((0,0) + t\overset{(\ell_1,\ell_2)}{\overset{\shortparallel}{\ell}}\right) - f(0,0)}{t}$$

$$= \lim_{t \to 0} \frac{f(t\ell_1, t\ell_2) - 0}{t} = \lim_{t \to 0} \frac{\sqrt{|(t\ell_1)^2 - (t\ell_2)^2|}}{t}$$

$$= \lim_{t \to 0} \frac{|t|\sqrt{|\ell_1^2 - \ell_2^2|}}{t}$$

To have the limit, we need to have $\lim\limits_{t \to 0^-} \cdots = \lim\limits_{t \to 0^+} \cdots$

$$\Rightarrow \sqrt{|\ell_1^2 - \ell_2^2|} = -\sqrt{|\ell_1^2 - \ell_2^2|} \Rightarrow |\ell_1^2 - \ell_2^2| = 0$$

$$\Rightarrow \ell_1 = \mp \ell_2$$

for $\ell = (a, -a) \in \mathbb{R}^2$ s.t. $a \in \mathbb{R}$ the directional derivative at $(0,0)$ exists. /

Problem 2.

Review. Arc length parametrization is a parametrization of a curve $\gamma : I \to \mathbb{R}^d$ such that $|\gamma'| = 1$.

For a curve with Arc length parametrization the Frenet frame or TNB frame are:

$$\tau := \gamma' \qquad n := \frac{\gamma''}{|\gamma''|} \qquad b := \tau \times n$$

Therefore, first you need to find a arc-length parametrization of the given curve. and then compute τ, n, b.

Ex 3. Find an arc-length parametrisation of the curve $\gamma : [0, \pi] \longrightarrow \mathbb{R}^4$

$$\gamma(t) = (\sin t, \sin 2t, \cos t, \cos 2t)$$

$$\gamma'(t) = (\cos t, 2\cos 2t, -\sin t, -2\sin 2t)$$

$$|\gamma'(t)| = \sqrt{(\cos t)^2 + 4(\cos 2t)^2 + (\sin t)^2 + 4(\sin 2t)^2} = \sqrt{5}$$

let $t = \frac{\alpha}{\sqrt{5}}$ $\quad \alpha \in [0, \sqrt{5}\pi]$

let $\quad \tilde{\gamma} : [0, \sqrt{5}\pi] \longrightarrow \mathbb{R}^4$

$$\tilde{\gamma}(\alpha) = \left(\sin\frac{\alpha}{\sqrt{5}}, \sin\left(\frac{2\alpha}{\sqrt{5}}\right), \cos\frac{\alpha}{\sqrt{5}}, \cos\left(\frac{2\alpha}{\sqrt{5}}\right)\right)$$

This is the same curve and

$$\tilde{\gamma}'(\alpha) = \left(\frac{1}{\sqrt{5}}\cos\frac{\alpha}{\sqrt{5}}, \frac{2}{\sqrt{5}}\cos\left(\frac{2\alpha}{\sqrt{5}}\right), \frac{-1}{\sqrt{5}}\sin\frac{\alpha}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\sin\left(\frac{2\alpha}{\sqrt{5}}\right)\right)$$

and

$$|\tilde{\gamma}'(\alpha)| = 1.$$

problem 3.

Thm 4.15. let $F: G \to \mathbb{R}^n$, $\underset{\text{open}}{G} \subset \mathbb{R}^{m+n}$,

F is of class C^1. i.e. differentiable with continuous differential. and rank of $\nabla f(z) = n$

$\forall z \in G$. Then

$$M := \{ z \in G \mid f(z) = 0 \}$$

is a differentiable manifold and $\forall z_0 \in M$

$$T_{z_0} M = \{ h \in \mathbb{R}^{m+n} \mid \nabla F(z_0) h = 0 \}$$

and

$$N_{z_0} M = \text{lin} \{ \nabla F_1(z_0), \dots, \nabla F_n(z_0) \}.$$

Ex 4. Find a tangent plane to the graph of the function $f(x,y) = x + y^2$, $(x,y) \in \mathbb{R}^2$ through the point $\underbrace{(1,-1,2)}_{P_0}$. Find the normal line as well.

Solution. $F(x,y,z) = x + y^2 - z$

$$M = \{ p \in \mathbb{R}^3 \text{ s.t. } F(p) = 0 \}$$

$$\nabla F(P_0) = \left[\frac{\partial F}{\partial x}(P_0) \quad \frac{\partial F}{\partial y}(P_0) \quad \frac{\partial F}{\partial z}(P_0) \right]$$

$$= \left[1 \quad -2 \quad -1 \right]$$

$$T_{P_0} M = \left\{ \underset{(h_1,h_2,h_3)}{h} \in \mathbb{R}^3 : [1 \ -2 \ -1] \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} = 0 \right\}$$

$\Rightarrow$ $T_{P_0} M$ is the solution of $h_1 - 2h_2 - h_3 = 0$

This is a plane in $\mathbb{R}^3$. (This is a subspace of dim $\underline{2}$.)

$$N_{P_0} M = \underset{t \in \mathbb{R}}{(1 \ -2 \ 1) \cdot t} \rightarrow \text{This is a line in } \mathbb{R}^3.)$$

Problem 4.

Review. Hessian: let $g: U \to \mathbb{R}$, $U \subseteq \mathbb{R}^n$, $x_0 \in U$

then $Hg(x_0) = \left[\frac{\partial^2 g}{\partial x_k \partial x_j} \right]_{kj}$ $\quad 1 \leq k, j \leq n$

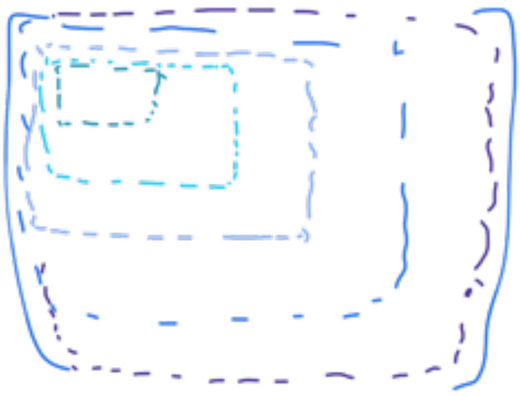
By Schwarz Theorem (4.16) we know that

$Hg(x_0)$ is symmetric since $\frac{\partial^2 g}{\partial x_k \partial x_j} = \frac{\partial^2 g}{\partial x_j \partial x_k}$

$\forall k, j \in \{1, \dots, n\}$

✓

A symmetric matrix is positive definite if the determinant of all the minors of the form



(i.e. the $k \times k$ matrices on the top-left corner for $k = 1, \dots, n$.) are positive.

It is negative definite if the sign alternates

$\boxed{-}\ \boxed{+}\ \boxed{-}\ \dots$

Thm 4.17. $g : U \to \mathbb{R}$, $x_0 \in U$,

if

- g is differentiable on U
- It's partial derivatives are differentiable at x_0
- $\text{grad}\, g(x_0) = 0$ (grad $g = \nabla g$ when $g : U \to \underline{\underline{\mathbb{R}}}$)

Then

- $Hg(x_0)$ Positive definite $\Rightarrow g(x_0)$ is a local min.
- " negative " $\Rightarrow$ " " " " max.

Ex 5. Find a local extrema of

$$f : \mathbb{R}^2 \to \mathbb{R}$$
$$(x, y) \mapsto x^2 + y^2$$

and

$$g : \mathbb{R}^2 \to \mathbb{R}$$
$$(x, y) \mapsto x^2 - y^2$$

Solution:

Both f and g are differentiable everywhere and their partial derivatives are differentiable everywhere.

$$\text{grad } f = \begin{bmatrix} 2x & 2y \end{bmatrix} = 0 \Rightarrow x = y = 0$$

$$Hf(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad 2 > 0 \qquad 4 > 0$$

$\rightarrow Hf$ is positive definite $\Rightarrow f(0,0)$ is a local minimum.

$$\text{grad } g = \begin{bmatrix} 2x & -2y \end{bmatrix} = 0 \Rightarrow x = y = 0$$

$$Hg(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

This is not positive or negative definite:

$g(0,0)$ is not a local extrema.

```
f = z - x ^ 2 - y ^ 2
k = 1.5;
ContourPlot3D[f == 0, {x, -k, k}, {y, -k, k}, {z, -k + 1 / 2, k + 1 / 2}]
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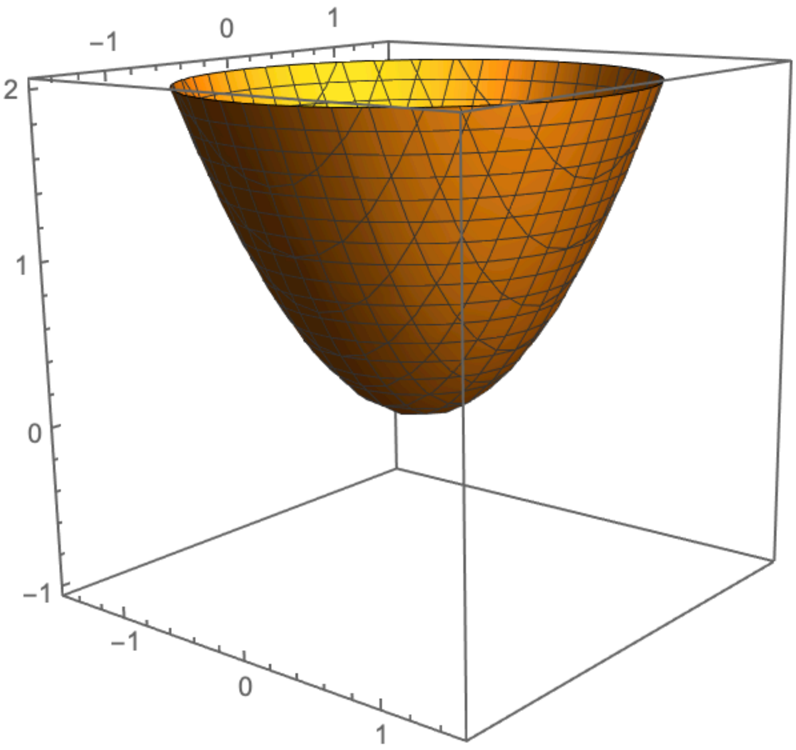
$-x^2 - y^2 + z$

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f = z - x ^ 2 + y ^ 2
k = 1.5;
ContourPlot3D[f == 0, {x, -k, k}, {y, -k, k}, {z, -k + 1 / 2, k + 1 / 2}]
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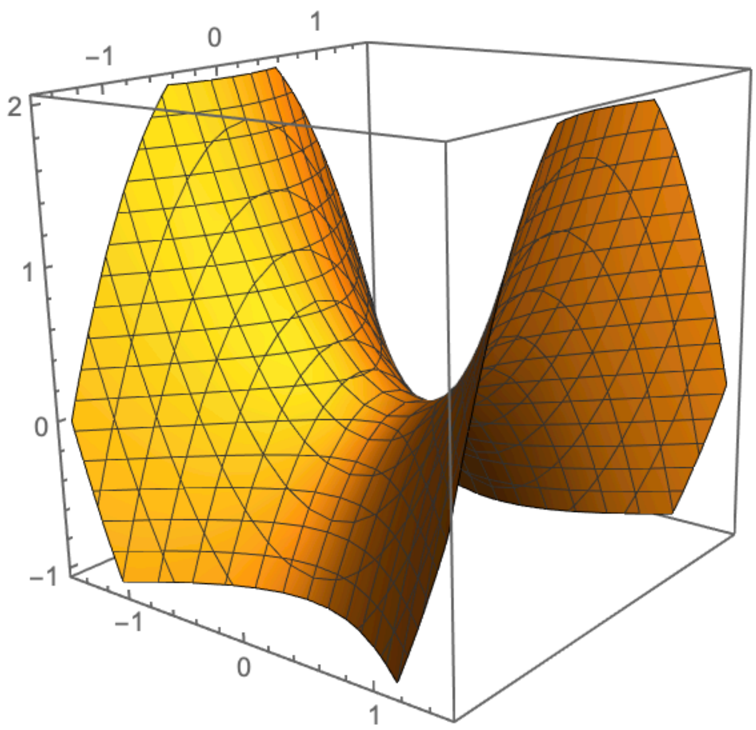
$-x^2 + y^2 + z$



Problem 5.

we use implicit function theorem.

- $F: G \to \mathbb{R}^n \qquad G \subseteq \mathbb{R}^{m+n}$
- F of class C^1 on G.
- $x_0 \in \mathbb{R}^m, y_0 \in \mathbb{R}^n, (x_0, y_0) \in G, F(x_0, y_0) = 0$
- $\det D_y F(x_0, y_0) \neq 0$

  * The coordinates of x_0 are <u>independent</u> around the point (x_0, y_0)
  * The coordinates of y_0 are <u>dependent</u> around the point (x_0, y_0)

- $D_y f$ is part of the matrix ∇f that is only the partial derivatives wrt coordinates of y. This is a $n \times n$ submatrix of the $n \times (m+n)$ matrix ∇f.

$$\nabla f = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} \cdots \frac{\partial f_1}{\partial x_m} & \frac{\partial f_1}{\partial y_1} \cdots \frac{\partial f_1}{\partial y_n} \\ \vdots & \\ \underbrace{\frac{\partial f_n}{\partial x_1} \cdots \frac{\partial f_n}{\partial x_m}}_{D_x f} & \underbrace{\frac{\partial f_n}{\partial y_1} \cdots \frac{\partial f_n}{\partial y_n}}_{D_y f} \end{bmatrix}$$

Then,

$\exists f : U \longrightarrow \mathbb{R}^n$ s.t. f is C^1 and

$f(x_0) = y_0$ and $F(x, f(x)) = 0 \quad \forall x \in U$

Although Implicit function thm does not give us the explicit formula for f, we can still compute its differential. (page 70)

$$\underset{n \times m}{\nabla f(x_0)} = - \underset{n \times n}{\Big(\underbrace{D_y F(x_0, y_0)} \Big)^{-1}} \underset{n \times m}{D_x F(x_0, y_0)}$$

Remember that this is a square matrix.

* You can always reorder the variables and move the dependent variables to the end.

Ex 6. Let $G: \mathbb{R}^2 \to \mathbb{R}$ s.t.

$$G(x,y) = x^2 - 3xy + y^3 - 7$$

Is there a function $f: U \to \mathbb{R}$, $3 \in U \subseteq \mathbb{R}$ s.t. $f(3) = 4$ and $\forall x_0 \in U$ $G(f(x_0), x_0) = 0$?

If yes, find the differential of f at 3.

Solution:

First observe that $G(4,3) = 0$

$$\frac{\partial G}{\partial x}(4,3) = 2x - 3y \Big|_{(4,3)} = -1 \neq 0$$

$$\frac{\partial G}{\partial y}(4,3) = -3x + 3y^2 \Big|_{(4,3)} = 15 \quad \underbrace{\neq 0}_{\text{not needed}}$$

Therefore, you can choose any of x or y to be the dependent. We choose x since the question is asking for it.

Thus, there is a function $f : U \longrightarrow \mathbb{R}$ $\quad U \subseteq \mathbb{R}$

s.t. $f(3) = 4$ and $\forall x_0 \in U$

$$G(f(x_0), x_0) = 0$$

and $$\partial f(3) = -\underbrace{\left(D_x G(4,3)\right)^{-1}}\left(D_y G(4,3)\right)$$

$$= -(-1)^{-1}(15) = \boxed{15}$$