Series 5

1. Use spherical coordinates

$$(x, y, z) \;=\; (r \sin\vartheta \cos\varphi, r \sin\vartheta \sin\varphi, r \cos\vartheta)\,, \quad \vartheta \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}],\ \varphi \in [0, 2\pi]$$

on $\mathbb{R}^3$.

a) Express the differential forms

$$\alpha = xdx + ydy + zdz,$$
$$\beta = xdy \wedge dz - ydx \wedge dz + zdx \wedge dy,$$
$$\gamma = \mathrm{vol} := dx \wedge dy \wedge dz$$

in spherical coordinates and dr, $d\vartheta$, $d\varphi$ in cartesian coordinates. *3 pts.*

b) Find a 2-form $0 \neq \omega \in \Omega^2(\mathbb{R}^3 \setminus \{0\})$ such that $d\omega = 0$ but there exists no $\eta \in \Omega^1(\mathbb{R}^3 \setminus \{0\})$ with $d\eta = \omega$. Compute $\int_{S^2(r_o)} \omega$, where $S^2_{r_o} = \{(x, y, z) \,|\, x^2 + y^2 + z^2 = r_o^2\}$, $r_o > 0$.
Hint: Use Ansatz $\omega = i_X \mathrm{vol}$ for a vector field $X = f(r)\frac{\partial}{\partial r}$, $f \in C^\infty(\mathbb{R}^3 \setminus \{0\}, \mathbb{R})$. *3 pts.*

2. Consider the 2-dimensional manifold

$$T_{r,R} \;=\; \{\, (x, y, z) \,|\, (\sqrt{x^2 + y^2} - R)^2 + z^2 = r^2 \,\}$$

for $0 < r < R$.

a) Why is $T_{r,R}$ a smooth manifold? *1 pt*

b) Find two 1-forms $\alpha, \beta \in \Omega^1(T_{r,R})$ s.t. $d\alpha = d\beta = 0$ but there ex. no $f, g \in C^\infty(T_{r,R})$ with $df = \alpha$, $dg = \beta$, and such that $\alpha \wedge \beta$ vanishes nowhere on $T_{r,R}$. *2 pts.*

c) Compute $\int_{T_{r,R}} \alpha \wedge \beta$ for your solutions. *1 pt.*

3. Let $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ with the coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^n) \in \mathbb{R}^n \times \mathbb{R}^n$ and with the 2-form $\omega_o \in \Omega^2(\mathbb{R}^{2n})$ given by

$$\omega_o \;=\; dx^1 \wedge dy^1 + \ldots + dx^n \wedge dy^n\,.$$

a) Show that ω_o induces an isomorphism for any open subset $U \subset \mathbb{R}^{2n}$

$$\mathcal{X}(U) \xrightarrow{\cong} \Omega^1(U),$$
$$X \mapsto i_X \omega_o\,.$$

1 pt.

b) Given $H \in C^\infty(U, \mathbb{R})$ define $X_H \in \mathcal{X}(U)$ by $i_{X_H}\omega_o = dH$. Write the formula for the ordinary differential equations of the local flow of X_H in the (x, y)-coordinates. *1 pt.*

Turn page!

c) Show that the formula

$$X_{\{f,g\}} = [X_f, X_g]$$

defines an $\mathbb{R}$-bilinear operation $\{\cdot,\cdot\}\colon C^\infty(U) \times C^\infty(U) \to C^\infty(U)$ uniquely up to a constant, and give an explicit expression for $\{f,g\}$ in the (x,y)-coordinates. *1 pt.*

d) Let ϕ_H^t be the local flow of the vectorfield X_H. Show that $\frac{d}{dt}(H \circ \phi_H^t) = 0$. *1 pt.*

4. Let Q be an arbitrary smooth n-dimensional manifold and consider its cotangent bundle T^*Q. If $(q^1, \ldots, q^n)$ are local coordinates on Q, then the canonically associated coordinates on T^*Q are $(q^1, \ldots, q^n, p_1, \ldots, p_n)$. Show that the formula $X = \sum_i p_i \frac{\partial}{\partial p_i}$ defines a globally well-defined vector field on T^*Q. *2 pts.*

5. *Optional Problem*: Let M be a smooth manifold and $\nabla\colon \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$, $(X,Y) \mapsto \nabla_X Y$ a connection. Then define the operator

$$R\colon \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$$

by

$$R(X,Y,Z) := R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\,.$$

Show that R is a 3-1-tensor field, i.e. it can be understood as a section of $\bigotimes^3 T^*M \otimes TM$. *4 pts.*

Hand-In: Practice Session Wednesday Nov. 20