University of Leipzig-SS2020
Analysis-2 Jan Burczak

Problem Sheet 9

Due Date: 27.04.2020, 12:00 UTC+2 (CEST)

if you see a contradiction then it is NOT solvable.

Problem 1. [**3** pts] Solve the following system of linear of equations.

1- fix the order
2- make the matrix
3- do the row operations until it is RREF.

$$\begin{cases} 3x_1 + 9x_2 - 2x_3 + 17x_4 - 13x_5 = 16 \\ 2x_1 + 7x_2 + 7x_4 - 2x_5 = 11 \\ 2x_1 + 5x_2 - 2x_3 + 13x_4 - 13x_5 = 11 \\ x_1 + 3x_2 - x_3 + 5x_4 - 4x_5 = 5 \end{cases} \tag{1}$$

4- write the simplified system.
5- solve it!

Problem 2. [**3** pts] Which values of $r \in \mathbb{R}$ make the following system solvable? Find these solutions.

do 1,2 and 3,

$$\begin{cases} x_1 + 2x_2 + 2x_3 + x_4 = 3 \\ 3x_1 + 8x_2 + 8x_3 + 7x_4 = 9 \\ 2x_1 + 5x_2 + 6x_3 + 5x_4 = 7 \\ rx_4 + x_1 + 3x_2 + 4x_3 = 5 \end{cases} \tag{2}$$

4- find that for which values of r you don't see a contradiction.
The system is solvable for those values.

Problem 3. [**2+2+1+3*** pts] Compute determinants of

2 simple ways to compute the determinant:

i) $A = \begin{bmatrix} 3 & 4 & 2 & 2 \\ 4 & 5 & 6 & 5 \\ 2 & 3 & 6 & 0 \\ 8 & 7 & 7 & 8 \end{bmatrix}$

1- do the row operations until you have the RREF of the matrix. Now you have an upper triangular matrix. The determinant is the multip. of the diagonals.

ii) $B = \begin{bmatrix} 3 & 6 & 0 & 6 & 3 \\ 4 & 5 & 0 & 4 & 2 \\ 5 & 4 & 3 & 3 & 2 \\ 4 & 3 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 & 1 \end{bmatrix}$

iii) $det(A^5B^2)$

2- expand with respect to a row/column: (or simply, follow the pattern:)

2x2 matrix: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$

3x3 matrix: $\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = (-1)^{\boxed{1+1}} a_{\boxed{11}} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} + (-1)^{\boxed{1+2}} a_{\boxed{12}} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$

$+ (-1)^{\boxed{1+3}} a_{\boxed{13}} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$

for iii- just note that $\det(M_1 M_2) = \det(M_1) \times \det(M_2)$

* for which n-tuples $(s_1, s_2, \cdots, s_n) \in \mathbb{R}^n$ the equation $\det A = 0$

where

$$A = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & s_1 - x & s_1 & \cdots & s_1 & s_1 \\ 1 & 1 & 2s_2 - x & \cdots & s_2 & s_2 \\ \vdots & & & & & \vdots \\ 1 & 1 & 1 & \cdots & 1 & ns_n - x \end{bmatrix}_{(n+1)\times(n+1)}$$

has a unique solution $x = 0$.

what does the prolem mean? let's set $s_i = 0$ for $\forall i \in \{1, \dots, n\}$

then we have

$$A = \begin{bmatrix} 1 & 1 & \cdots & \cdots & 1 \\ 1 & -x & 0 & \cdots & 0 \\ 1 & 1 & -x & \cdots & 0 \\ & & \cdots & & \\ 1 & 1 & \cdots & 1 & -x \end{bmatrix}.$$

what is $\det A$?

take $r_i = r_i - r_1 \quad \forall i \geq 2$.

$$\det A = \det \begin{bmatrix} 1 & 1 & \cdots & \cdots & 1 \\ 0 & -x-1 & 1 & \cdots & 1 \\ & & -x-1 & & \vdots \\ & 0 & & \ddots & 1 \\ & & & & -x-1 \end{bmatrix} = (-x-1)^n \longrightarrow$$

so $\det A = 0$ has a unique solution $x = -1$

but we want this solation to be $x = 0$

If we do the same process without setting $s_i = 0 \ \forall i$

we get

$$\det A = \det \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & s_1 - 1 - x & s_1 - 1 & \cdots & s_1 - 1 \\ & & 2s_2 - 1 - x & \cdots & s_2 - 1 \\ & 0 & & \ddots & \vdots \\ & & & & ns_n - 1 - x \end{bmatrix} =$$

$$\overbrace{(s_1 - 1 - x)}^{0}\overbrace{(2s_2 - 1 - x)}^{0} \cdots \overbrace{(ns_n - 1 - x)}^{0} = 0$$

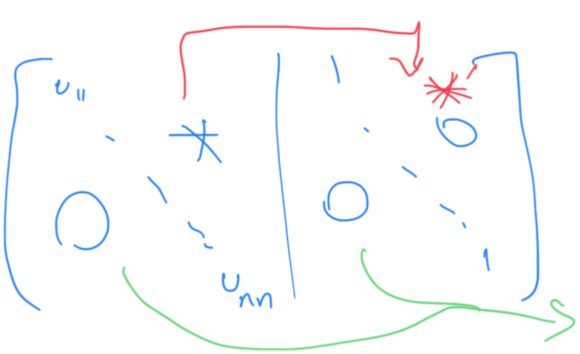
$$\Longrightarrow (s_1, \dots, s_n) = \left(1, \frac{1}{2}, \dots, \frac{1}{n}\right) \in \mathbb{R}^n$$

* Prove that the determinant of a diagonal matrix (if exists) D is diagonal. What is the relation between D_{ii} and D^{-1}_{ii}.

$$\left[\begin{array}{c|c} D_{n\times n} & I_{n\times n} \end{array}\right]_{n\times 2n} = \left[\begin{array}{cccc|cccc} d_{11} & & O & & 1 & 0 & \cdots & 0 \\ & d_{22} & & & 0 & 1 & \cdots & 0 \\ O & & \ddots & & 0 & 0 & \ddots & \vdots \\ & & & d_{nn} & 0 & \cdots & & 0 \end{array}\right]$$

we make the RREF of D: devide the first row by d_{11}.
then you need to use 1 to make the rest of the first column zero. but the rest is already zero. So you move to the second row, the first nonzero element is d_{22}. devide everything by d_{22}. now the same story again, ...

If instead of D it was an Upper-triangular U



do you see that these zeros never change?

Problem 5. [**2+2+2** pts] Invert matrices

If A^{-1} exists then $Ax=b$ has a unique sol. $x = A^{-1}b$.

If not just solve $Ax=b$ as in problem $\underline{1}$.

how to invert a matrix $A_{n\times n}$:

$$[A \mid I]_{n\times 2n} \xrightarrow[\text{of } A]{\text{make RREF}} \left[\begin{array}{c|c} \text{RREF of } A & B \end{array}\right]$$

if this is $I_{n\times n}$ then $B = A^{-1}$

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ 0 & 3 & 2 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 3 & 1 & 1 \\ 2 & -1 & -2 & 2 \\ -1 & -2 & 2 & -1 \\ 3 & 1 & -1 & 2 \end{bmatrix}$$

for b_2 you can already use the RREF of A, do you see why?

and solve (if possible) $A_i x = b_i$, $i = 1, 2$ with A_i above and $b_1 = \begin{bmatrix} 3 \\ 5 \\ 0 \\ 6 \end{bmatrix}$, $b_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Problem 6. [**3** pts] What are the values of parameters $s, t \in \mathbb{R}$ such that

$$c_1 = (5, 7, s, 2), c_2 = (1, 3, 2, 1), c_3 = (2, 2, 4, t)$$

of $\mathbb{R}^4$ are linearly independent?

linear independency means that non of the vectors can be written as a linear combination of the other two. In another words the RREF of the matrix

$$\begin{array}{cc} \begin{bmatrix} 5 & 7 & s & 2 \\ 1 & 3 & 2 & 1 \\ 2 & 2 & 4 & t \end{bmatrix} & \begin{array}{c} \leftarrow c_1 \\ \leftarrow c_2 \\ \leftarrow c_3 \end{array} \end{array}$$

does not have a zero row.

So just form the RREF and check for which s,t no row is zero. (If you have heard of the "Rank", you can say the rank of the matrix has to be $\underline{3}$.)

The Gram–Schmidt process [edit]

We define the projection operator by

$$\mathrm{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle}\mathbf{u},$$

where $\langle \mathbf{u}, \mathbf{v} \rangle$ denotes the inner product of the vectors **u** and **v**. This operator projects the vector **v** orthogonally onto the line spanned by vector **u**. If **u** = **0**, we define $\mathrm{proj}_{\mathbf{0}}(\mathbf{v}) := \mathbf{0}$. i.e., the projection map $\mathrm{proj}_{\mathbf{0}}$ is the zero map, sending every vector to the zero vector.

The Gram–Schmidt process then works as follows:

for you $v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ $v_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

$v_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

$$\begin{aligned}
\mathbf{u}_1 &= \mathbf{v}_1, & \mathbf{e}_1 &= \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} \\
\mathbf{u}_2 &= \mathbf{v}_2 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_2), & \mathbf{e}_2 &= \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} \\
\mathbf{u}_3 &= \mathbf{v}_3 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_3) - \mathrm{proj}_{\mathbf{u}_2}(\mathbf{v}_3), & \mathbf{e}_3 &= \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} \\
\mathbf{u}_4 &= \mathbf{v}_4 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_4) - \mathrm{proj}_{\mathbf{u}_2}(\mathbf{v}_4) - \mathrm{proj}_{\mathbf{u}_3}(\mathbf{v}_4), & \mathbf{e}_4 &= \frac{\mathbf{u}_4}{\|\mathbf{u}_4\|} \\
&\vdots & &\vdots \\
\mathbf{u}_k &= \mathbf{v}_k - \sum_{j=1}^{k-1} \mathrm{proj}_{\mathbf{u}_j}(\mathbf{v}_k), & \mathbf{e}_k &= \frac{\mathbf{u}_k}{\|\mathbf{u}_k\|}.
\end{aligned}$$

The sequence $\mathbf{u}_1$, ..., $\mathbf{u}_k$ is the required system of orthogonal vectors, and the normalized vectors $\mathbf{e}_1$, ..., $\mathbf{e}_k$ form an ortho*normal* set. The calculation of the sequence $\mathbf{u}_1$, ..., $\mathbf{u}_k$ is known as *Gram–Schmidt orthogonalization*, while the calculation of the sequence $\mathbf{e}_1$, ..., $\mathbf{e}_k$ is known as *Gram–Schmidt orthonormalization* as the vectors are normalized.

Example [edit]

Euclidean space [edit]

Consider the following set of vectors in $\mathbf{R}^2$ (with the conventional inner product)

$$S = \left\{ \mathbf{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\}.$$

Now, perform Gram–Schmidt, to obtain an orthogonal set of vectors:

$$= \frac{\left\langle \begin{pmatrix} 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\rangle}{\left\langle \begin{pmatrix} 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\rangle} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \frac{6+2}{9+1} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\mathbf{u}_1 = \mathbf{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\mathbf{u}_2 = \mathbf{v}_2 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_2) = \begin{pmatrix} 2 \\ 2 \end{pmatrix} - \mathrm{proj}_{\left(\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right)} \left(\begin{pmatrix} 2 \\ 2 \end{pmatrix} \right) = \begin{pmatrix} 2 \\ 2 \end{pmatrix} - \frac{8}{10} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -2/5 \\ 6/5 \end{pmatrix}.$$

We check that the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are indeed orthogonal:

$$\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = \left\langle \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} -2/5 \\ 6/5 \end{pmatrix} \right\rangle = -\frac{6}{5} + \frac{6}{5} = 0,$$

noting that if the dot product of two vectors is *0* then they are orthogonal.

For non-zero vectors, we can then normalize the vectors by dividing out their sizes as shown above:

$$\mathbf{e}_1 = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ 1 \end{pmatrix} \qquad \sqrt{3^2+1^2} = \sqrt{10}$$

$$\mathbf{e}_2 = \frac{1}{\sqrt{\frac{40}{25}}} \begin{pmatrix} -2/5 \\ 6/5 \end{pmatrix} = \frac{1}{\sqrt{10}} \begin{pmatrix} -1 \\ 3 \end{pmatrix}.$$

Preparatory Exercises for the Problem Sheet 1

SS2020 - Analysis 2 - University of Leipzig

Mahsa Sayyary Namin

First week

In all of the following problems we are working over the field of real numbers. (Note that the equation $x^2+1=0$ has no solution over $\mathbb{R}$, however, it has two solutions $\pm i$ over $\mathbb{C}$)

Problem 1. Determine whether the following systems of equations have any solutions? If no, explain why, and if yes write all the solutions.

$$\begin{cases} 2x+y=0, \\ y+x=1, \\ 2x-y=0. \end{cases}$$

$$\begin{cases} 2x_1-3x_2-7x_3+5x_4+2x_5=-2, \\ x_1-2x_2-4x_3+3x_4+x_5=-2, \\ 2x_1-4x_2+x_5+2x_4=3, \\ x_1-5x_2-7x_3+6x_4+2x_5=-7 \end{cases}$$

Solution 1. The first system has no solution because adding up the first and the last equations result in

$$\begin{aligned} 4x=0 &\Rightarrow x=0 \\ &\Rightarrow y=0 && (since\ 2x+y=0) \\ &\Rightarrow 1=0 && (since\ x+y=0) \end{aligned}$$

r_i corresponds to the i-th equation.

solve it, we can form the reduced row echelon form by doing row operations:

$$\begin{array}{c} \\ r_1 \\ r_2 \\ r_3 \\ r_4 \end{array}\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & \end{array} \\ \left(\begin{array}{ccccc|c} 2 & -3 & -7 & 5 & 2 & -2 \\ 1 & -2 & -4 & 3 & 1 & -2 \\ 2 & 0 & -4 & 2 & 1 & 3 \\ 1 & -5 & -7 & 6 & 2 & -7 \end{array}\right) \end{array} \xrightarrow{r_1 \longleftrightarrow r_2} \begin{array}{c} r_2 \\ r_1 \\ r_3 \\ r_4 \end{array}\left(\begin{array}{ccccc|c} 1 & -2 & -4 & 3 & 1 & -2 \\ 2 & -3 & -7 & 5 & 2 & -2 \\ 2 & 0 & -4 & 2 & 1 & 3 \\ 1 & -5 & -7 & 6 & 2 & -7 \end{array}\right)$$

$$\xrightarrow{\substack{r_1 = r_1 - 2r_2 \\ r_3 = r_3 - 2r_2 \\ r_4 = r_4 - r_2}} \begin{array}{c} r_2 \\ r_1 \\ r_3 \\ r_4 \end{array}\left(\begin{array}{ccccc|c} 1 & -2 & -4 & 3 & 1 & -2 \\ 0 & 1 & 1 & -1 & 0 & 2 \\ 0 & 4 & 4 & -4 & -1 & 7 \\ 0 & -3 & -3 & 3 & 1 & -5 \end{array}\right)$$

$$\xrightarrow{\substack{r_2 = r_2 + 2r_1 \\ r_3 = r_3 - 4r_1 \\ r_4 = r_4 + 3r_1}} \begin{array}{c} r_2 \\ r_1 \\ r_3 \\ r_4 \end{array}\left(\begin{array}{ccccc|c} 1 & 0 & -2 & 1 & 1 & 2 \\ 0 & 1 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{r_3 = -r_3} \begin{array}{c} r_2 \\ r_1 \\ r_3 \\ r_4 \end{array}\left(\begin{array}{ccccc|c} 1 & 0 & -2 & 1 & 1 & 2 \\ 0 & 1 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow{r_4 = r_4 - r_3} \begin{array}{c} r_2 \\ r_1 \\ r_3 \\ r_4 \end{array}\left(\begin{array}{ccccc|c} 1 & 0 & -2 & 1 & 1 & 2 \\ 0 & 1 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The simplified version of the original system of equations is:

$$\begin{cases} x_1 - 2x_3 + x_4 = 1 \\ x_2 + x_3 - x_4 = 2 \\ x_5 = 1 \end{cases}$$

By setting $x_3 = \alpha$ and $x_4 = \beta$, the solution set is

$$\{(2\alpha - \beta + 1, -\alpha + \beta + 2, \alpha, \beta, 1) \in \mathbb{R}^5 :\ \alpha, \beta \in \mathbb{R})\}$$

Problem 2. Let $A = \begin{pmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, and $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$. For which values of b the system $Ax = b$ has a solution?

Solution 2. We start with simplifying the system as in the previous problem:

$$\begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array} \left(\begin{array}{ccc|c} 3 & -1 & 2 & b_1 \\ 2 & 1 & 1 & b_2 \\ 1 & -1 & 1 & b_3 \end{array}\right) \xrightarrow{r_1 \longleftrightarrow r_3} \begin{array}{c} r_3 \\ r_2 \\ r_1 \end{array} \left(\begin{array}{ccc|c} 1 & -1 & 1 & b_3 \\ 2 & 1 & 1 & b_2 \\ 3 & -1 & 2 & b_1 \end{array}\right)$$

$$\xrightarrow[r_1 = r_1 - 3r_3]{r_2 = r_2 - 2r_3} \begin{array}{c} r_3 \\ r_2 \\ r_1 \end{array} \left(\begin{array}{ccc|c} 1 & -2 & 1 & b_3 \\ 0 & 5 & -1 & b_2 - 2b_3 \\ 0 & 5 & -1 & b_2 - 3b_3 \end{array}\right) \xrightarrow[r_1 = r_1 - r_2]{} \begin{array}{c} r_3 \\ r_2 \\ r_1 \end{array} \left(\begin{array}{ccc|c} 1 & -2 & 1 & b_3 \\ 0 & 5 & -1 & b_2 - 2b_3 \\ 0 & 0 & 0 & b_1 + b_2 - 5b_3 \end{array}\right)$$

for $\{b \in \mathbb{R}^3 : b_1 + b_2 = 5b_3\}$, the system $Ax = b$ has (at least) one solution.

Problem 3. Find the determinant of the following matrices:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 4 & 8 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ -1 & 0 & 3 & \dots & n-1 & n \\ -1 & -2 & 0 & \dots & n-1 & n \\ & & & & & \\ \dots & \dots & \dots & \dots & \dots & \dots \\ & & & & & \\ -1 & -2 & -3 & \dots & -(n-1) & 0 \end{pmatrix}$$

Remark 1. Based on your questions in the class: Determinant is ONLY defined for square matrices.

Solution 3. The first equation shows how to find the determinant of a 2×2 matrix. The second one shows how to reduce the problem of finding a 3×3 determinant to the 2×2 case. The third equation reduces finding a 4×4 determinant to 3×3 determinants. Therefore you can see that following this pattern you can see how to find a $n\times n$ determinant by finding some number of 2×2 determinants.

3 is in the row $\boxed{1}$ and column $\boxed{3}$. $(-1)^{\boxed{1+3}} = +1$ therefore the sign is Possitive

$$\det A = +1\times 4 - 2\times 3 = -2$$

$$\det B = +1\times \det\begin{pmatrix} 2 & 1\\ 4 & 8\end{pmatrix} - 2\times \det\begin{pmatrix} 3 & 1\\ 2 & 8\end{pmatrix} + 3\times \det\begin{pmatrix} 3 & 2\\ 2 & 4\end{pmatrix} = \dots$$

here we expand with respect to the last row because of the cancellations..

$$\det C = -1\times\begin{pmatrix} 2 & 3 & 4\\ 3 & 2 & 1\\ 1 & 0 & 1\end{pmatrix} + \underbrace{0\times\begin{pmatrix} 1 & 3 & 4\\ 4 & 2 & 1\\ 0 & 0 & 1\end{pmatrix}}_{0} - 1\times\begin{pmatrix} 1 & 2 & 4\\ 4 & 3 & 1\\ 0 & 1 & 1\end{pmatrix}$$

$$+\underbrace{0\times\begin{pmatrix} 1 & 2 & 3\\ 4 & 3 & 2\\ 0 & 1 & 0\end{pmatrix}}_{0} = \dots$$

Note that:

- Determinant of a triangular matrix is the product of the elements on the diagonal,
- Row and column operations do not change the determinant.

By adding the first row of the matrix D to all the other rows you get the triangular matrix:

$$\begin{pmatrix} 1 & 2 & 3 & \dots & n-1 & n\\ 0 & 2 & \star & \dots & \star & \star\\ 0 & 0 & 3 & \star & \star & \star\\ & & & & & \\ \dots & \dots & \dots & \dots & \dots & \dots\\ & & & & & \\ 0 & 0 & 0 & \dots & 0 & n\end{pmatrix}$$

the star elements are not important.

Therefore $\det D = 1\times 2\times 3\times\dots n = n!$

Problem 4. Let A and B be two $n \times n$ matrices. If $\det A = 5$ and $detB = 7$, compute $\det(A^2 B^3 A)$.

Solution 4.

$$\begin{aligned}\det(A^4 B^3 A) &= \det(A^5 B^3) \\ &= \det(A^5) \times \det(B^3) \\ &= (\det A)^5 \times (\det B)^3 \\ &= 5^5 \times 7^3.\end{aligned}$$

Note that:

- If A and B are two matrices such that AB and BA are both defined, then $\det AB = \det BA$,
- If A and B are two matrices of size $n \times n$ then $\det AB = \det A \times \det B$

Problem 5. Let $A = \begin{pmatrix} a & 0 & 0 \\ d & b & 0 \\ f & e & c \end{pmatrix}$ such that a, b, and c are nonzero, compute A^{-1}.

Remark 2. Based on your questions in the class: Inverse is ONLY defined for square matrices with nonzero determinant.

Solution 5. In this problem we show that the inverse of a lower triangular 3×3 matrix (if exists) is lower triangular and the entries on the diagonal of A^{-1} are the inverse of the corresponding entries on the diagonal of A. Moreover you can see a general method to find the inverse of a matrix.

Since none of a, b, or c are zero, $\det A = abc \neq 0$ and the inverse A^{-1} exists. We attach an identity matrix of the proper size to A and make the reduced row echelon form of A that is going to be the identity matrix. The identity matrix that we attached at the beginning will turn to A^{-1} by this process.

$$\begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array}\left(\begin{array}{ccc|ccc} a & 0 & 0 & 1 & 0 & 0 \\ d & b & 0 & 0 & 1 & 0 \\ f & e & c & 0 & 0 & 1 \end{array}\right) \xrightarrow{r_1 = r_1/a} \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{a} & 0 & 0 \\ d & b & 0 & 0 & 1 & 0 \\ f & e & c & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{r_2 = r_2 - d.r_1,\; r_3 = r_3 - f.r_1} \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{a} & 0 & 0 \\ 0 & b & 0 & \frac{-d}{a} & 1 & 0 \\ 0 & e & c & \frac{-f}{a} & 0 & 1 \end{array}\right)$$

$$\xrightarrow{r_2 = r_2/b} \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{a} & 0 & 0 \\ 0 & 1 & 0 & \frac{-d}{ab} & \frac{1}{b} & 0 \\ 0 & e & c & \frac{-f}{a} & 0 & 1 \end{array}\right)$$

$$\xrightarrow{r_3 = r_3 - e.r_2} \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{a} & 0 & 0 \\ 0 & 1 & 0 & \frac{-d}{ab} & \frac{1}{b} & 0 \\ 0 & 0 & c & \frac{-f}{a} + \frac{de}{ab} & \frac{-e}{b} & 1 \end{array}\right) \xrightarrow{r_3 = r_3/c} \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{a} & 0 & 0 \\ 0 & 1 & 0 & \frac{-d}{ab} & \frac{1}{b} & 0 \\ 0 & 0 & 1 & \frac{-f}{ac} + \frac{de}{abc} & \frac{-e}{bc} & \frac{1}{c} \end{array}\right)$$

$$\text{Therefore } A^{-1} = \begin{pmatrix} \frac{1}{a} & 0 & 0 \\ \frac{-d}{ab} & \frac{1}{b} & 0 \\ \frac{-f}{ac} + \frac{de}{abc} & \frac{-e}{bc} & \frac{1}{c} \end{pmatrix}.$$

Problem 6. Let A be the matrix from the previous problem. Does the system $Ax = b$ have a solution? Is the solution unique?

Solution 6. Yes, it has a solution and it is unique. Let $v = A^{-1}b$ then $A(A^{-1}b) = b$. Thus v is a solution. Now let w be a solution. We have $Aw = b$. Multiplying both sides by A^{-1} gives $w = A^{-1}b = v$. Therefore the solution is unique.

Remark 3. If the matrix A is invertible then the system $Ax = b$ has a unique solution. If the matrix A is not invertible the system can either have no solution or infinity many solutions. We have discussed both cases in the class.

Problem 7. For which values of α and β, the vectors $v_1 = (\alpha, 1)$ and $v_2 = (0, \beta)$ form a basis for $\mathbb{R}^2$. In other words, v_1 and v_2 are linearly independent.

Solution 7. Two vectors v_1 and v_2 forming a basis for $\mathbb{R}^2$ is equivalent to $\det \begin{pmatrix} \alpha & 1 \\ 0 & \beta \end{pmatrix} \neq 0$. Therefore, we need $\alpha \neq 0$ and $\beta \neq 0$.

Problem 8. Use the alternating symbol (Levi-Civita symbol) ϵ_{ijk} and the Kronecker delta δ_{ij} to show

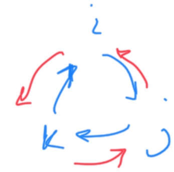
$$u.(v \times w) = v.(w \times u) = w.(u \times v).$$

Solution 8. To be discussed in our next meeting.

$e_i \quad e_j \quad e_k \rightarrow$ canonical basis

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

$$\varepsilon_{ijk} = \begin{cases} \pm 1 & \\ 0 & i=j \text{ or } i=k \text{ or } j=k \end{cases}$$



NOTE that the inner product of two vectors A, B is

$$A \cdot B = \overbrace{A_i \, \delta_{ij} \, B_j}^{\text{sum over } i,j} \quad \text{and the outer product:}$$

$$(A \times B)_k = \overbrace{A_i \, B_j \, \varepsilon_{ijk}}^{\text{sum over } i,j}$$

Now: $C \cdot (A \times B) = C_i \, \delta_{ij} \, (A \times B)_j = C_i \, \delta_{ij} \, (\overbrace{\varepsilon_{mnj} \, A_m}^{\varepsilon_{mni} = \varepsilon_{nim}} B_n)$

$$= A_m \left(\varepsilon_{nim} \, B_n \, C_i \right) = A_m \, \delta_{mj} \left(\varepsilon_{nij} \, B_n \, C_i \right) =$$

$$A_m \, \delta_{mj} \, (B \times C)_j = A \cdot (B \times C).$$

$\blacksquare$