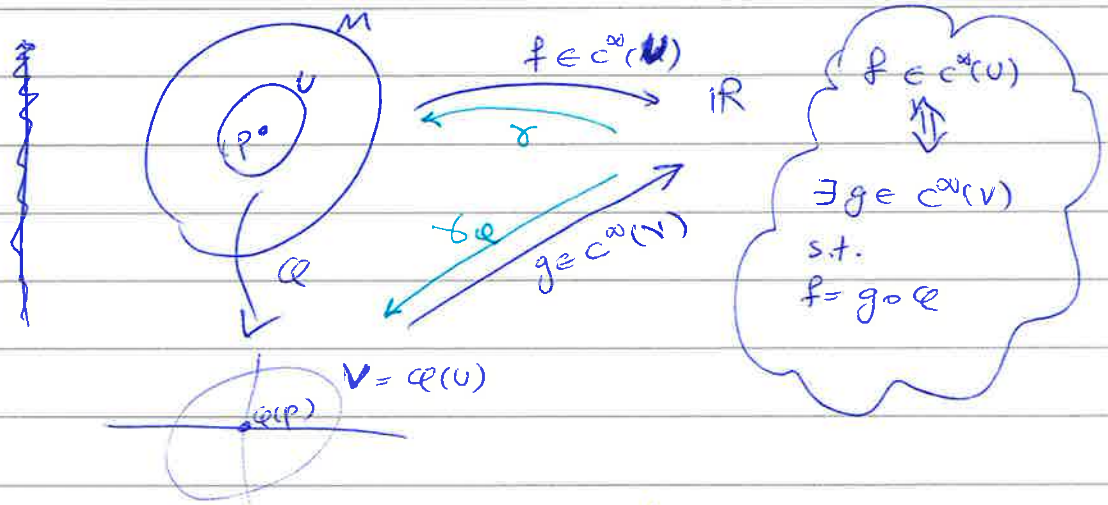
Problem sheet 3- 3.1

DEF: Let $X \in \mathfrak{X}(U)$, $U \subset \mathbb{R}^n$, $\gamma : (a,b) \to U$, $\gamma \in C^1((a,b), \mathbb{R}^n)$

γ is called an integral curve for $X \iff \dot{\gamma}(t) = \frac{d}{dt}(\gamma(t)) = X(\gamma(t))$ $\forall t \in (a,b)$.

3.1. $T_pM := \{ [\gamma]_p \mid \gamma : (-\varepsilon, \varepsilon) \to M, \gamma(0) = p \}$ $\quad \gamma_1 \sim \gamma_2 \iff \forall f \in C^\infty(M)\ \frac{d}{dt}\Big|_{t=0}(f \circ \gamma_1) = \frac{d}{dt}\Big|_{t=0}(f \circ \gamma_2)$

$\tilde{T}_pM := \{ [X]_p \mid X \in \mathfrak{X}(M) \}$ $\quad X \sim Y \iff X(f)(p) = Y(f)(p)$



let (U, V, φ) be a chart. We need to find a derivation $X_\varphi \in \mathfrak{X}(V)$ which corresponds to a derivation $X \in \mathfrak{X}(U)$ i.e. we want that

$X_\varphi(g)(\varphi(p)) = X(g \circ \varphi)(p) \quad \forall g \in C^\infty(V)$

Therefore X_φ is defined as $X_\varphi(g) = (X(g \circ \varphi)) \circ \varphi^{-1}$. Check that $X_\varphi \in C^\infty(V)$ i.e. check that $\forall g_1, g_2 \in C^\infty(V)$:

$$X_\varphi(g_1 g_2) = g_1 X_\varphi(g_2) + g_2 X_\varphi(g_1). \checkmark$$

let γ_φ be the integral curve for X_φ : $\gamma_\varphi : (-\varepsilon, \varepsilon) \to V$, $0 \longmapsto \varphi(p)$

(*) $X_\varphi(\gamma_\varphi(t)) = \frac{d}{dt}(\gamma_\varphi(t))$ $\forall t \in (-\varepsilon, \varepsilon)$ (in particular in 0) ← initial condition

define $\gamma : (-\varepsilon, \varepsilon) \to M$ to be $\gamma = \varphi^{-1} \circ \gamma_\varphi$.

The claim is that the map $\psi_1 : \tilde{T}_pM \to T_pM$ that takes $X \in \mathfrak{X}(U)$ to γ as constructed above, is well-defined, one to one and onto, also indep. of choice of chart.

$$\left(\frac{d}{dt}\Big|_{t=0}(\varphi^1\circ\gamma), \ldots, \frac{d}{dt}\Big|_{t=0}(\varphi^n\circ\gamma)\right)$$

From (*) $X_\varphi(\varphi(p)) = \frac{d}{dt}\Big|_{t=0} (\varphi\circ\gamma)(t)$ (**)

But $X_\varphi = \sum a^i_\varphi \frac{\partial}{\partial x_i}$ where $a^i \in C^\infty(V)$ and $X_\varphi(\pi_i) = a^i_\varphi$

$$X_\varphi(\varphi(p)) = (a^1(\varphi(p)), \ldots, a^n(\varphi(p)))$$ ← remember.

$\forall x \in V$: $a^i_\varphi(x) = X_\varphi(\pi_i) = (X(\pi_i\circ\varphi))\circ\varphi^{-1}(x) = X(\varphi^i)\circ\varphi^{-1}(x)$ ($\pi_i\circ\varphi = \varphi^i$)

$\varphi = (\varphi^1, \ldots, \varphi^n)$ now let $x = \varphi(p)$

$\Rightarrow a^i_\varphi(\varphi(p)) = X(\varphi^i)(p)$

from (**) $X(\varphi^i)(p) = \frac{d}{dt}\Big|_{t=0} (\varphi^i\circ\gamma)(t)$

well def: let $X = Y$ i.e. $\forall f \in C^\infty(M)$ $X(f)(p) = Y(f)(p)$
and let $\psi_1(X) = \gamma_1$ and $\psi_1(Y) = \gamma_2$ WTS $\gamma_1 = \gamma_2$
$\Rightarrow$ in particular $X(\varphi^i)(p) = Y(\varphi^i)(p) \Rightarrow$
$\forall i \in \{1, \ldots, n\}$

$$\frac{d}{dt}\Big|_{t=0} (\varphi\circ\gamma_1) = \frac{d}{dt}\Big|_{t=0} (\varphi\circ\gamma_2)$$ and $\gamma_1(0) = \gamma_2(0) = p$ ← this is important.

this implies $\frac{d}{dt}\Big|_{t=0} g\circ\varphi\circ\gamma_1 = \frac{d}{dt}\Big|_{t=0} g\circ\varphi\circ\gamma_2$ $\forall g \in C^\infty(V)$

$\Rightarrow \frac{d}{dt}\Big|_{t=0} f\circ\gamma_1 = \frac{d}{dt}\Big|_{t=0} f\circ\gamma_2$ $\forall f \in C^\infty(U)$ $\Rightarrow \gamma_1 = \gamma_2$ ✓

1-1 $\gamma_1 = \gamma_2 \Rightarrow X(\varphi^i)(p) = Y(\varphi^i)(p)$ $i = 1, \ldots, n$

$\Rightarrow X_\varphi(\varphi(p)) = Y_\varphi(\varphi(p)) \Rightarrow a^i(\varphi(p)) = b^i(\varphi(p)) \Rightarrow X_\varphi(g)(\varphi(p)) = Y_\varphi(g)(\varphi(p))$ $\forall g \in C^\infty(V)$

($X_\varphi = \sum a^i \frac{\partial}{\partial x^i}$, $Y_\varphi = \sum b^i \frac{\partial}{\partial x_i}$)

$\Rightarrow X(f)(p) = Y(f)(p)$ $\forall f \in C^\infty(U)$ $\Rightarrow X = Y$ ✓

being onto just follows from the definition of Ψ_1.

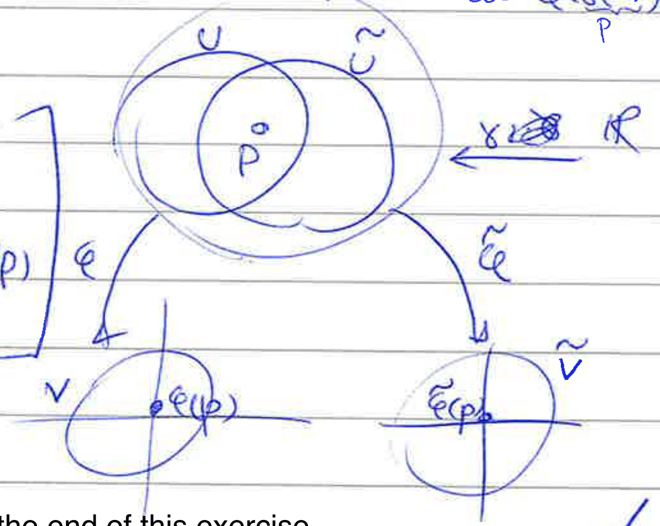
now to show that this is indep. of the chart show that if $(\tilde{U},\tilde{V},\tilde{\varphi})$ is another chart $p\in\tilde{U}$, then

(i) $\frac{d}{dt}\Big|_{t=0} \varphi\circ\gamma = J\,\frac{d}{dt}\Big|_{t=0} \tilde{\varphi}\circ\gamma$ $\Phi_{ij} = J$ = Jacobian of $\tilde{\varphi}\circ\varphi^{-1}$ at $\varphi(\underbrace{\gamma(0)}_{p})$

and

$$\begin{bmatrix} X(\varphi^1)(p) \\ \vdots \\ X(\varphi^n)(p) \end{bmatrix} = J \begin{bmatrix} X(\tilde{\varphi}^1)(p) \\ \vdots \\ X(\tilde{\varphi}^n)(p) \end{bmatrix}$$

(ii)



*** To see more details on this check the end of this exercise

$T_pM = \{[\gamma]_p : \gamma:(-\varepsilon,\varepsilon)\to M,\ \gamma(0)=p\}$ $\gamma_1\sim\gamma_2 \Leftrightarrow \forall f\in C^\infty(M)$ $\frac{d}{dt}\Big|_{t=0}(f\circ\gamma_1) = \frac{d}{dt}\Big|_{t=0}(f\circ\gamma_2)$

$\tilde{T}_pM = \{(v_i)_{i\in I_p}\in(\mathbb{R}^n)^{I_p} \mid v_i = \Phi_{ij}(v_j)\ \ \forall i,j\in I_p\}$

$\Phi_{ij} = D(\varphi_i\circ\varphi_j^{-1})(\varphi_j(p))$ $p\in U_i\cap U_j$

$\Psi_2 : T_pM \to \tilde{T}_pM$

$\gamma \mapsto \left(\frac{d}{dt}\Big|_{t=0}\varphi_i\circ\gamma\right)_{i\in I_p}$

by the def. of $[\gamma]_p$ and (i), Ψ_2 is well-defined.

$\boxed{1\text{-}1}$ $\left(\frac{d}{dt}\Big|_{t=0} \varphi_i \circ \gamma_1\right)_{i\in I_p} = \left(\frac{d}{dt}\Big|_{t=0} \varphi_i \circ \gamma_2\right)_{i\in I_p}$

$\underset{(U_i, V_i, \varphi_i)}{\Rightarrow} \forall i \in I_p \quad \frac{d}{dt}\Big|_{t=0} \varphi_i \circ \gamma_1 = \frac{d}{dt}\Big|_{t=0} \varphi_i \circ \gamma_2$

$\Rightarrow \forall g \in C^\infty(V_i) \quad \frac{d}{dt}\Big|_{t=0} g \circ \varphi_i \circ \gamma_1 = \frac{d}{dt}\Big|_{t=0} g \circ \varphi_i \circ \gamma_2$

$\Rightarrow \forall f \in C^\infty(U_i) \quad \frac{d}{dt}\Big|_{t=0} f \circ \gamma_1 = \frac{d}{dt}\Big|_{t=0} f \circ \gamma_2$

$\Rightarrow \gamma_1 = \gamma_2$ ✓

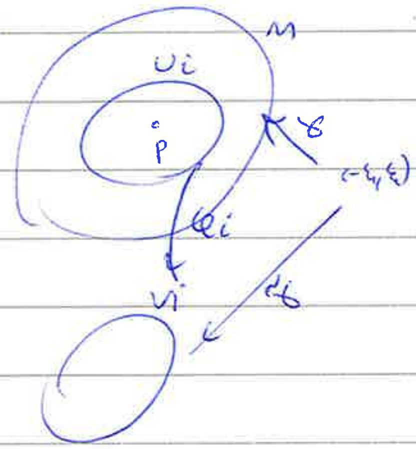
onto

let $v_i \in V_i = \varphi_i(U_i)$ WTS $\exists \gamma : (-\varepsilon, \varepsilon) \to U_i$ s.t.

$v_i = \frac{d}{dt}\Big|_{t=0} \varphi_i \circ \gamma$

Let $\tilde{\gamma}(t) = t \cdot v_i + x_0$ for some $x_0 \in V_0$ s.t. $\forall t \in (-\varepsilon, \varepsilon)$ $\tilde{\gamma}(t) \in V_i$ we have $\frac{d}{dt}\Big|_{t=0} \tilde{\gamma}(t) = v_i$

$\tilde{\gamma} : (-\varepsilon, \varepsilon) \to V_i$



Now define $\gamma : (-\varepsilon, \varepsilon) \to U_i$ to be

$\gamma = \varphi_i^{-1} \circ \tilde{\gamma} \Rightarrow \tilde{\gamma} = \varphi_i \circ \gamma$

and $\frac{d}{dt}\Big|_{t=0} \varphi_i \circ \gamma = v_i$ /

$\Psi_3 : \tilde{T}_p M \to T_p M$

$(v_i)_{i\in I_p} \longmapsto [X]_p$ s.t. $X(\varphi_i^j)(p) = v_i^j \quad \forall j = 1, \dots, n$

(no matter which $i \in I_p$ we choose, this is well-def. bcz of (ii)

and $\overbrace{(v_i)_{i\in I_p}}^{a} = \overbrace{(w_i)_{i\in I_p}}^{b} \Rightarrow \Psi_3(a) = \Psi_3(b)$ follows by the same argument as 1-1 for Ψ_1. /

$(v_i^{\varphi V_i})_{i\in I_p} \longrightarrow X \qquad (w_i^{\varphi V_i})_{i\in I_p} \longrightarrow Y$

1-1 $X = Y \Rightarrow X(f)(p) = Y(f)(p) \quad \forall f \in C^\infty(M)$

in part. $X(\varphi_i^j)(p) = Y(\varphi_i^j)(p) \Rightarrow v_i = w_i$

Onto like in (**) we know that $X(\varphi_i^j)(p) = a^j_{\varphi_i}(\varphi_i(p))$

let $a^j_{\varphi_i} = v_i^j$ constant function, $X_{\varphi_i} = \sum a^i_{\varphi_i} \frac{\partial}{\partial x_i}$

and $X_{\varphi_i}(g) = X(g\circ\varphi_i)\circ\varphi_i^{-1} \quad g \in C^\infty(V_i)$.

Thus we have

$$\begin{cases} \psi_1 : \bar{T}_pM \to T_pM, \; [X]_p \to [\gamma]_p \text{ s.t. } X(\varphi^i)(p) = \frac{d}{dt}\Big|_{t=0} \varphi^i\circ\gamma \\ \psi_2 : T_pM \to \tilde{T}_pM, \; \gamma \to \left(\frac{d}{dt}\Big|_{t=0} \varphi_i\circ\gamma\right)_{i\in I_p} \\ \psi_3 : \tilde{T}_pM \to \bar{T}_pM, \; (v_i)_{i\in P} \longrightarrow X \text{ s.t. } X(\varphi_i^j)(p) = v_i^j \end{cases}$$

$\psi_3 \circ \psi_2 \circ \psi_1 = id$

*** To verify this we have to check how both sides transform under change of coordinate chart:

Let: x^i be the coordinates for a chart φ and y^i be the coordinates for a chart $\tilde{\varphi}$

Left side: $\frac{d}{dt}\big|_{t=0}\gamma_{\tilde{\varphi}^i} = \frac{d}{dt}\big|_{t=0}\tilde{\varphi}^i \circ \gamma = \frac{d}{dt}\big|_{t=0}(\tilde{\varphi}^i \circ \varphi^{-1}) \circ (\varphi \circ \gamma) = \frac{d}{dt}\big|_{t=0}(y^i(x)) \circ x(\gamma)$

$= \sum_j \frac{\partial y^i}{\partial x^j} \frac{d}{dt}\big|_{t=0}\varphi^j \circ \gamma = \sum_j J^i_j \frac{d}{dt}\big|_{t=0}\gamma_{\varphi^j}$ Where J is the jacobian of the transformation function $\tilde{\varphi} \circ \varphi^{-1}$ taken at $\varphi(\gamma(0))$

Right side: $X(\tilde{\varphi}^i) = X(\tilde{\varphi}^i \circ \varphi^{-1} \circ \varphi)(p) = X(\tilde{\varphi}^i \circ \varphi^{-1} \circ \varphi)\varphi^{-1}(\underbrace{\varphi(p)}_{=x_0}) =$

$X(\pi_i \circ (\tilde{\varphi} \circ \varphi^{-1}) \circ \varphi)\varphi^{-1}(x_0) = X_\varphi(\pi_i \circ (\tilde{\varphi} \circ \varphi^{-1}))(x_0) = \sum_j \frac{\partial y^i}{\partial x^j} a^j_\varphi(x_0) = \sum_j J^i_j X(\varphi^j)(p)$

Using the vector notation we have

$\frac{d}{dt}\Big|_{t=0} \gamma_{\tilde{\varphi}} = J(\frac{d}{dt}\Big|_{t=0} \gamma_\varphi)$ and

$X(\tilde{\varphi})(p) = JX(\varphi)(p)$