Problem sheet 9.

1. Lemma (parallelogram-identity)

(i) $\forall v, w, u \in H$: $\|v+w\|^2 + \|v-w\|^2 = 2\|v\|^2 + 2\|w\|^2$

(ii) $\|v-w\|^2 = 2\|v-u\|^2 + 2\|w-u\|^2 - 4\|\frac{1}{2}(v+w)-u\|^2$

Proof. (i) $\|v+w\|^2 + \|v-w\|^2 = \langle v+w, v+w\rangle + \langle v-w, v-w\rangle$

$= 2\langle v,v\rangle + 2\langle w,w\rangle + \cancel{2Re\langle v,w\rangle} - \cancel{2Re\langle v,w\rangle} = 2\|v\|^2 + 2\|w\|^2$ /

(ii) Use (i) and replace v and w with $v-u$ and $w-u$ respectively:

$\|v-u+w-u\|^2 + \|v-u-w+\cancel{u}\|^2 = 2\|v-u\|^2 + 2\|w-u\|^2$

$\Rightarrow \|v-w\|^2 = 2\|v-u\|^2 + 2\|w-u\|^2 - 4\|\frac{1}{2}(v+w)-u\|^2$ /

Proof of the exercise: $\forall x \in H\setminus C$ define $d_x := \inf\{\|x-y\| \mid y \in C\}$

Then $\forall x \in H\setminus C \overset{WTS}{\exists!} y \in C$ s.t. $d_x = \|x-y\|$.

By definition of d_x, $\exists (y_n)_{n\in\mathbb{N}} \subset C$ for fixed $x \in H\setminus C$ s.t.

$\|x-y_n\| \longrightarrow d_x$. This sequence is Cauchy because:

$\|y_n - y_m\|^2 = 2\|x-y_n\|^2 + 2\|x-y_m\|^2 - 4\|\frac{1}{2}(y_n+y_m)-x\|^2$

where $\|x-y_n\|^2 \leq d_x^2 + \xi^2$ and $\|x-y_m\|^2 \leq d_x^2 + \xi^2$ for some $\xi>0$.

By convexity of C $\frac{1}{2}(y_n+y_m) \in C$, therefore $-4\|\frac{1}{2}(y_n+y_m)-x\|^2 \leq -4d^2$

$\Rightarrow \|y_n - y_m\|^2 \leq 4\xi^2 \longrightarrow 0$ as $\xi \to 0$

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Since C is complete $(y_n)_{n\in\mathbb{N}}$ converges to some $y \in C$ with $\|x-y\| = d_x$. /

Uniqueness: suppose $z \in C$ s.t. $\|x-z\| = d = \|x-y\|$

$$\rightarrow 0 \le \|y-z\|^2 \overset{(ii)}{=} 2\underbrace{\|y-x\|^2}_{d^2} + 2\underbrace{\|z-x\|^2}_{d^2} - 4\underbrace{\left\|\tfrac{1}{2}(y+z)-x\right\|^2}_{\le -4d^2 \text{ as before}} \le 0$$

$$\Rightarrow \|y-z\| = 0 \Rightarrow y = z.$$

$\blacksquare$

2-a) First we show that $V^\perp$ is closed. Then we use problem (1) to show that any element x of H can be uniquely written as $x = y_1 + z_1$ where $y_1 \in V$ and $z_1 \in V^\perp$. For that it is enough to show $V \cap V^\perp = \{0\}$ and any element x can be written as $\underset{V}{x = y_1} + \underset{V^\perp}{z_1}$.

$\boxed{V^\perp \text{ is closed}}$ let $(x_n) \in V^\perp$ with $\lim_{n\to\infty} x_n = x_\infty$. and $y \in V$:

$$0 = \langle \lim_{n\to\infty}(x_n - x_\infty), y\rangle \underset{\langle\,,\,\rangle \text{ is continuous.}}{=} \lim_{n\to\infty}\langle x_n - x_\infty, y\rangle = \lim_{n\to\infty}\langle x_n, y\rangle - \langle x_\infty, y\rangle$$

$$= 0 - \langle x_\infty, y\rangle \Rightarrow \langle x_\infty, y\rangle = 0 \Rightarrow x_\infty \in V^\perp. /$$

(It is also easy to see that $V^\perp$ is convex but you actually do not need that).

$\boxed{V \cap V^\perp = \{0\}}$: Assume $x \in V$ and $x \in V^\perp$ then $\langle x, x\rangle = 0 \Rightarrow x = 0$

$\Rightarrow V \cap V^\perp = \{0\}$. /

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Now, for $x \in H$ we want to find $y_1 \in V$ and $z_1 \in V^\perp$ s.t. $x = y_1 + z_1$.

If $x \in V$ then $x = \underset{\in V}{x} + \underset{\in V^\perp}{0}$ and we are done. Assume $x \notin V$. Since V is closed $\exists!\, y_1 \in V$ s.t. $\|x - y_1\| = \inf\{\|x-y\| \mid y \in V\}$.

Define

$$f(t) = \|x - (y_1 + t y')\|^2$$

for an arbitrary $y' \in V$.

Because of the linearity of the inner product that induces the norm we can differentiate for t and because we have chosen y_1 in a way that it minimizes $f(t)$, we expect that $f'(t) = 0$ for $t = 0$.

$$0 = \frac{d}{dt}\langle (x-y_1) - ty', (x-y_1) - ty' \rangle\Big|_{t=0} = -2\,\mathrm{Re}\,\langle x - y_1, y' \rangle$$

Now set:

$$g(t) = \|x - (y_1 + t i y')\|^2 \quad \text{for the same } y' \in V.$$

$\Longrightarrow$

$$0 = \frac{d}{dt}\langle (x-y_1) - ity', (x-y_1) - ity' \rangle\Big|_{t=0} = +2\,\mathrm{Im}\,\langle x - y_1, y' \rangle$$

Since $\mathrm{Re}\,\langle x-y_1, y'\rangle = 0$ and $\mathrm{Im}\,\langle x - y_1, y'\rangle = 0$ we have

$\langle x - y_1, y' \rangle = 0$ for an arbitrary choice of $y' \in V$. Thus we have $x - y_1 \in V^\perp$. Set $z_1 = x - y_1$. Now we are done since

$x = \underset{\in V}{y_1} + \underset{\in V^\perp}{z_1}$.

(2.b) Consider $T: H \to H^*$, $T(v) = \langle v, \cdot \rangle$. first we want to show that

$$\| T(v) \|_{op} = \|v\| \quad \forall v \in H.$$

by definition of $\| \ \|_{op}$ (from your very first lectures and exercises):

$$\| T(v) \|_{op} = \sup_{\substack{\|w\|=1 \\ w \in H}} \| T(v)(w) \| = \sup_{\substack{\|w\|=1 \\ w \in H}} \| \langle v, w \rangle \| \quad (*)$$

from Cauchy-Schwartz inequality

$$\| \langle v, w \rangle \| \leq \|v\| \cdot \overbrace{\|w\|}^{1} = \|v\| \xRightarrow{\text{taking sup}} \| T(v) \|_{op} \leq \|v\|.$$

On the other hand let $w = \frac{v}{\|v\|}$. from $(*)$

$$\underbrace{\| \langle v, \frac{v}{\|v\|} \rangle \|} \leq \sup_{\substack{\|w\|=1 \\ w \in H}} \| \langle v, w \rangle \| = \| T(v) \|_{op}$$

$$\hookrightarrow = \frac{1}{\|v\|} \| \langle v, v \rangle \| = \frac{\|v\|^2}{\|v\|} = \|v\|.$$

$$\Rightarrow \|v\| \leq \| T(v) \|_{op} \quad \checkmark$$

Note that $\langle \lambda v, \mu w \rangle = \bar{\lambda} \mu \langle v, w \rangle$ for $\lambda, \mu \in \mathbb{C}$ therefore $T(\lambda v) = \langle \lambda v, \cdot \rangle = \bar{\lambda} \langle v, \cdot \rangle$ and thus T is not linear over $\mathbb{C}$. So you can only prove the isomorphism over $\mathbb{R}$:

T is linear over $\mathbb{R}$, so it remains to show that T is a bijection:

$\boxed{\text{T is injective:}}$ Let $T(v)=0$ WTS $v=0$.

$T(v)=0 \Rightarrow \langle v, w\rangle = 0 \quad \forall w \in H \Rightarrow v=0.$ /

$\boxed{\text{T is surjective}}$ $T: H \longrightarrow H^*$ $T(v)=\langle v, \cdot\rangle$

WTS $\forall f \in H^*$ (i.e. $f: H \longrightarrow \mathbb{R}$ linear) $\exists v \in H$

s.t. $T(v)=f$ (i.e. $\langle v, h\rangle = f(h) \quad \forall h \in H$)

If $f=0$ then $\forall h \in H$ $f(h)=0 \Rightarrow f=T(0)$. /

If $f \neq 0$ then let $V := \ker f$. V is a closed subspace of H. Since $f\neq 0$, $V \neq H \Rightarrow V^\perp \neq \{0\}$.

Let $t \in V^\perp$ s.t. $\|t\|=1$ and $f(t)\neq 0$. we claim

$f(h)=\langle v, h\rangle \quad \forall h \in H$ where $v=f(t)\cdot t$

Proof:

$\forall h \in H$ we have that $f\left(h-\frac{f(h)}{f(t)}t\right)=f(h)-\frac{f(h)}{\cancel{f(t)}}\cancel{f(t)}=0$

$\Rightarrow h-\frac{f(h)}{f(t)}t \in \ker f$ note that $\underbrace{t}_{\in V^\perp}.\underbrace{f(t)}_{\text{scalar.}} \in V^\perp$

$\Rightarrow \langle t.f(t), h-\frac{f(h)}{f(t)}t\rangle = 0 \quad \forall h \in H$

$\Rightarrow \langle t f(t), h\rangle = \langle t f(t), \frac{f(h)}{f(t)}t\rangle = f(h)\underbrace{\|t\|^2}_{1}$
$= f(h) \quad \forall h \in H.$

$\Rightarrow f(h)=\langle v, h\rangle \quad \forall h \in H$ where $v=f(t).t$. /

3. a) If we know that D is dense in $C^0([0,1])$ wrt L^2 norm (1)
and $C^0([0,1])$ ~ ~ ~ $L^2([0,1])$ ~ ~ ~. (2)
then we conclude that D ~ ~ ~ $C^0([0,1])$ ~ ~ ~.
which is what we have to show.

To have (1) we need to show that ~~...~~ density wrt $\|\cdot\|_\infty$ implies density wrt L^2-norm:
It is sufficient to show that for $(f_n)_{n\in\mathbb{N}}$ a sequence in D if $f_n \xrightarrow{n\to\infty} f \in C^0([0,1])$ wrt the sup norm then f_n also converges to f wrt $\| \ \|_{L_2}$.

$$\|f_n - f\|_{L_2}^2 = \int_0^1 (f_n(x)-f(x))^2\,dx \le \left(\sup_{x\in[0,1]} |f_n(x)-f(x)|\right)^2 \cdot 1$$
$$= \| \cdot \|_\infty^2 .$$

(2) has been given to you in your Lectures.

Now for the second part we WTS that for $f\in C^0([0,1])$

$$\int_0^1 f(x)x^n\,dx = 0 \quad \forall n\in\mathbb{N} \implies f\equiv 0$$

(see next page)

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3) a) part 2) $\forall f \in C^0([0,1])$: $\int_0^1 f(x) x^n dx = 0 \;\; \forall n \in \mathbb{N} \implies f \equiv 0$

Since D is dense in $C^0([0,1])$ and $f \in C^0([0,1])$, wrt $\|\cdot\|_\infty$ there exists a sequence $\{(P_n)_{n \in \mathbb{N}}\}$ of polynomials (in D) s.t.

$\lim_{n \to \infty} P_n = f$ uniformly.

Also $\int_0^1 f(x) x^n dx = 0 \;\; \forall n \in \mathbb{N} \implies \int_0^1 f(x) P_n(x) dx = 0 \;\; \forall n \in \mathbb{N}$

(* Note that $P_n(x) = a_0 + a_1 x + \dots + a_n x^n$ where $0 \leq i \leq n$ $a_i \in \mathbb{R}$ and $a_i < \infty$)

Thus $\lim_{n \to \infty} \int_0^1 f(x) P_n(x) dx = 0 \overset{(*)}{\implies}$ Proof at the end

$$\int_0^1 f(x) \lim_{n \to \infty} P_n(x) = 0 \implies \int_0^1 f^2(x) dx = 0 \implies f \equiv 0$$

Proof of $(*)$:

Let $\{(f_n)_{n \in \mathbb{N}}\}$ be a sequence of functions converging uniformly on a compact set K

$f_n \to f$. We have:

$\forall \varepsilon > 0 \;\; \exists N \in \mathbb{N}$ s.t. $\sup_K |f(x) - f_n(x)| < \varepsilon / \int_K 1 dx$ for $n > N$

Therefore $\left| \int_K f(x) dx - \int_K f_n(x) dx \right| = \left| \int_K (f - f_n)(x) \right|$

$$\leq \int_K |f(x) - f_n(x)| dx \leq \int_K \frac{\varepsilon}{\int_K 1 \, dx} dx = \varepsilon.$$

Thus $\int_K f_n(x) dx \to \int_K f(x) dx$ as $n \to \infty$.

(3.b) Gram-Schmidt:

$$e_n = \frac{f_n - \sum_{i=0}^{n-1} \langle e_i , f_n \rangle e_i}{\left\| f_n - \sum_{i=0}^{n-1} \langle e_i , f_n \rangle e_i \right\|} \qquad \forall n \geq 0$$

$$e_0 = \frac{f_0}{\|f_0\|} = 1$$

$$e_1 = \frac{x - \langle x, 1 \rangle \cdot 1}{\| x - \langle x, 1 \rangle \cdot 1 \|} \qquad \langle x, 1 \rangle = \int_{[0,1]} x\,dx = \frac{1}{2}$$

$$\left\| x - \frac{1}{2} \right\| = \left(\int_{[0,1]} \left(x - \frac{1}{2}\right)^2 dx \right)^{1/2} = \frac{1}{2\sqrt{3}}$$

$$\Rightarrow e_1 = \sqrt{3}\,(2x - 1)$$

$$e_2 = \frac{x^2 - \langle x^2, 1 \rangle - \langle x^2, \sqrt{3}(x - \frac{1}{2}) \rangle \cdot e_1}{\| x^2 - \langle x^2, 1 \rangle - \langle x^2, \sqrt{3}(x - \frac{1}{2}) \rangle \cdot e_1 \|} = ---$$

$$= \sqrt{5}\,(6x^2 - 6x + 1)$$

(please always use a computer to check your computations.)
(for example mathematica)

Now from (a) $\{e_0, e_1, e_2, \dots\}$ makes a basis for the hilbert space.

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4.a) WTS $\hat{f'}(k) = 2\pi i k \hat{f}(k) \quad \forall k \in \mathbb{R}$

$$\hat{f'}(k) = \int_{\mathbb{R}} e^{-2\pi i k x} f'(x)\,dx$$

Integral by part

$$= e^{-2\pi i k x} f(x) \Big]_{-\infty}^{\infty} - \int_{\mathbb{R}} f(x)(-2\pi i k) e^{-2\pi i k x}\,dx$$

this is zero because $\lim_{|x| \to \infty} |f(x)| = 0$

$$\Rightarrow \quad = 2\pi i k \int_{\mathbb{R}} e^{-2\pi i k x} f(x)\,dx = 2\pi i k \hat{f}(k)$$

4.b) YES!

the reason is that this condition $\lim_{|x| \to \infty} |f(x)| = 0$

can be concluded from the fact that $f, f' \in L^1(\mathbb{R})$.

Proof: we want to show that $f \in C^1 \cap L^1$ and $f' \in L^1$
then $\lim_{x \to \infty} f(x) = 0$

If $f \in L^1(\mathbb{R})$ then there exists a sequence $x_n \to \infty$ s.t.

$f(x_n) \to 0. \Rightarrow |f(x) - f(x_n)| \le \int_{x_n}^{x} |f'|\,dx$ for some $x > x_n$

$$\le \int_{x_n}^{\infty} |f'|\,dx \to 0$$

as $x_n \to \infty$

Since $\|f'\|_{L^1} < \infty$.

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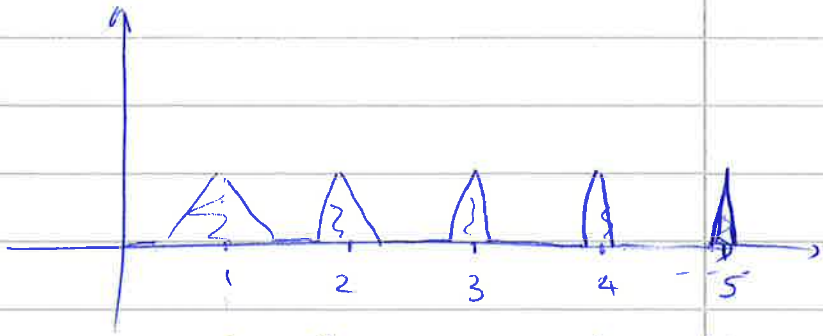
be careful that $f \in L^1(\mathbb{R})$ is not enough.

here is a counterexample



sum of the volume is $< \infty$ but $\lim_{n \to \infty} f(n)$ does not exist!

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Good luck! :)

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